

114. Asymptotic Equivalence in a Dynamical System

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1. Introduction. Let X be a metric space with its metric d . A dynamical system on X [1, p. 5] is defined to be an ordered triple (X, R, π) consisting of X , the real line R and a map $\pi: X \times R \rightarrow X$ such that:

- (a) $\pi(x, 0) = x$ for any $x \in X$,
- (b) $\pi(\pi(x, s), t) = \pi(x, s+t)$ for any $x \in X$ and all $s, t \in R$,
- (c) π is continuous on $X \times R$.

Given a dynamical system on X , the space X is called the phase space of the dynamical system.

An equivalence relation J on (X, R, π) is said to be invariant if $(x, y) \in J$ implies $(\pi(x, t), \pi(y, t)) \in J$ for any $t \in R$.

A large amount of research of the invariant equivalence relations in the phase space of dynamical systems has been done (e.g., see [2], [3], or [4]). However, the main concern of these is either the case in which the invariant equivalence relation is closed, or the case in which the phase space is compact.

In this paper we introduce an invariant equivalence relation, i.e., "*asymptotic equivalence*", which is neither closed nor the phase space compact, and then investigate the possibility of the construction of the quotient dynamical system induced by the equivalence relation. Main results obtained are Theorem 3.3 (a necessary and sufficient condition for the canonical surjection to be open) and Theorem 3.5 which gives a necessary and sufficient condition for the phase space of the quotient dynamical system to be Hausdorff.

2. Asymptotic equivalence.

Definition 2.1. If

$$d(\pi(x, t), \pi(y, t)) \rightarrow 0 \quad (t \rightarrow +\infty),$$

then x is said to be *asymptotically equivalent* to y , which is denoted xAy .

Remark 2.2. It is clear that the asymptotic equivalence A on (X, R, π) is an invariant equivalence relation.

Remark 2.3. The asymptotic equivalence A on (X, R, π) is not a closed relation, since we have a counterexample [1, p. 68, 2.4]: a dynamical system defined on R^2 by the differential equations

$$\frac{dx}{dt} = f(x, y), \quad \frac{dy}{dt} = g(x, y)$$

where

$$g(x, y) = -y \quad \text{for all } (x, y) \in R^2, \quad \text{and}$$

$$f(x, y) = \begin{cases} x & \text{if } x^2y^2 \geq 1 \\ 2x^3y^2 - x & \text{if } x^2y^2 < 1. \end{cases}$$

It is easily verified that for any $a > 0$ the point $((a, a^{-1}), (-a, a^{-1}))$ is an accumulation point of A which does not belong to A . Thus A is not closed.

3. Quotient dynamical system. Let X/A be the quotient set obtained by partitioning the phase space X of (X, R, π) by A . We assign to X/A the quotient topology relative to the canonical surjection $p: X \rightarrow X/A$, so that X/A becomes the quotient space relative to p . Each point of X/A can be represented by $p(x)$, where x is some point in X . We define a map

$$\rho: (X/A) \times R \rightarrow X/A$$

by the equality

$$\rho(p(x), t) = p(\pi(x, t)).$$

If ρ satisfies the axioms (a), (b) and (c) mentioned in the Introduction, then $(X/A, R, \rho)$ will become a dynamical system on X/A , which we call the quotient dynamical system induced by (X, R, π) .

Theorem 3.1. *If the canonical surjection p of X onto X/A is open, then $(X/A, R, \rho)$ is the quotient dynamical system induced by (X, R, π) .*

Proof. Since A is invariant, the validity of the theorem is a direct consequence of [3, p. 4, item 3 in article 1.5].

Notation 3.2. $B(x)$ denotes the set of $y \in X$ such that yAx .

The following theorem determines the class of dynamical systems for which the canonical surjections are open.

Theorem 3.3. *The canonical surjection p of X onto X/A is open if and only if any open subset S of X satisfies the following conditions:*

- (a) *if $x \in S$, then $B(x)$ is open or a border set in X ,*
- (b) *if $B(x)$ is a border set in X such that $B(x) \cap S \neq \phi$, then*

$$B(x) \subset \text{Int}(\cup \{B(y); y \in S\}),$$

where $\text{Int}(D)$ denotes the interior of a set D .

To prove the theorem we use the following lemma, which is well known [5, p. 97, Theorem 10]:

Lemma 3.4. *The canonical surjection p of X onto X/A is open if and only if $\cup \{B(x); x \in S\}$ is open for any S which is open in X .*

The proof of Theorem 3.3. Assume that p is open. Take any point $x \in S$. If $\text{Int}(B(x))$ is empty, then $B(x)$ is a border set. If $\text{Int}(B(x))$ is not empty, then

$$B(x) = \cup \{B(y); y \in \text{Int}(B(x))\}$$

is open in X by Lemma 3.4. Thus we have proved (a). Now we shall prove (b). Let $B(x)$ be a border set such that $B(x) \cap S \neq \phi$. There exists $y \in S$ such that $B(x) = B(y)$. Hence

$$B(x) \subset \cup \{B(z); z \in S\},$$

where $\cup \{B(z); z \in S\}$ is open by Lemma 3.4. Thus we have proved (b). Now assume (a) and (b). Let S be any open subset of X . The set $\cup \{B(z); z \in S\}$ is denoted T for convenience's sake. Clearly $B(y) \subset T$ for any $y \in T$, where $B(y)$ is open or a border set in X . Let $B(y)$ be a border set in X . There exists $u \in S$ such that $B(u) = B(y)$, so that $B(y) \subset \text{Int}(T)$ by the assumption (b). Here we define two sets T_b and T_o as follows:

$$\begin{aligned} T_b &= \cup \{B(x); x \in S \text{ and } B(x) \text{ is a border set in } X\}, \\ T_o &= \cup \{B(x); x \in S \text{ and } B(x) \text{ is open in } X\}. \end{aligned}$$

Then

$$T = T_b \cup T_o \subset \text{Int}(T) \cup \text{Int}(T) = \text{Int}(T),$$

so that T is open in X . Thus p is open by Lemma 3.4. Q.E.D.

The phase space of the quotient dynamical system is not necessarily Hausdorff, although X is a metric space. We give here a necessary and sufficient condition for the phase space of the quotient dynamical system to be Hausdorff.

Theorem 3.5. *Let X be a connected metric space, and let p be the open canonical surjection of X onto X/A . Then, X/A is Hausdorff if and only if*

(a) $X/A = \{X\}$

or

(b) $Bd(G) \subset G_b$

holds. Here

$$G = \cup \{B(x) \times B(x); x \in X\},$$

$$G_b = \cup \{B(x) \times B(x); x \in X \text{ and } B(x) \text{ is a border set in } X\},$$

and $Bd(G)$ is the boundary of G .

Proof. Assume that X/A is Hausdorff. Then G is closed in $X \times X$ [5, Theorem 11, p. 98]. Since p is open by the assumption, $B(x)$ is open or a border set for any $x \in X$ by Theorem 3.3. If $B(x)$ is open for any $x \in X$, then G is open in $X \times X$, so that $G = X \times X$ by the connectedness of $X \times X$. Hence X/A is the singleton $\{X\}$. Now we assume that there exists a $B(x)$ which is a border set in X . The set

$$G_o = \cup \{B(x) \times B(x); x \in X \text{ and } B(x) \text{ is open in } X\}$$

is open in $X \times X$, so that $G_o \subset \text{Int}(G)$. On the other hand,

$$G = G_b \cup G_o = Bd(G) \cup \text{Int}(G),$$

since G is closed. Moreover, $G_b \cap G_o = \phi$, and $\text{Int}(G) \cap Bd(G) = \phi$. Hence $Bd(G) \subset G_b$. Conversely, assume that (a) or (b) holds. If (a) is valid, then

$$G = \cup \{B(x) \times B(x); x \in X\} = X \times X,$$

so that G is closed. This fact and openness of p imply that X/A is Hausdorff [5, Theorem 11, p. 98]. Now assume that (b) holds. Then

$$\bar{G} = Bd(G) \cup Int(G) \subset G_b \cup Int(G) \subset G,$$

which implies that G is closed in $X \times X$. Consequently X/A is Hausdorff [5, Theorem 11, p. 98]. Q.E.D.

In connection with the Theorem 3.5 we can establish a necessary condition for X/A to be Hausdorff.

Theorem 3.6. *Let X be a connected metric space. Let the canonical surjection p of X onto X/A be open. If X/A is Hausdorff but is not a singleton, then every $B(x)$ is a border set in X .*

To prove Theorem 3.6 we need the following lemma:

Lemma 3.7. *Let $B(x)$ be open in X . If $u \in Bd(B(x))$, then*

$$(u, y) \in Bd(B(x) \times B(x))$$

for any $y \in B(x)$.

Proof. Let y be any point in $B(x)$, and let u be any point in $Bd(B(x))$. Clearly

$$(u, y) \bar{\in} B(x) \times B(x),$$

so that (u, y) is not an interior point of $B(x) \times B(x)$. Now assume that (u, y) is an exterior point of $B(x) \times B(x)$. Then, there exist neighborhoods U_1 and U_2 of u and y respectively such that

$$(U_1 \times U_2) \cap (B(x) \times B(x)) = \phi. \quad \dots (*)$$

Hence

$$(U_1 \cap B(x)) \times (U_2 \cap B(x)) = (U_1 \times U_2) \cap (B(x) \times B(x)) = \phi.$$

Here $U_1 \cap B(x) \neq \phi$ and $U_2 \cap B(x) \neq \phi$, since $u \in Bd(B(x))$ and $y \in B(x)$. Consequently

$(U_1 \times U_2) \cap (B(x) \times B(x)) \neq \phi$, which contradicts (*), so that (u, y) is not an exterior point of $B(x) \times B(x)$. Thus

$$(u, y) \in Bd(B(x) \times B(x)).$$

Q.E.D.

Proof of Theorem 3.6. Since p is open, every $B(x)$ is a border set or open in X . Assume that there exists $x \in X$ such that $B(x)$ is open. Then, $Bd(B(x))$ is not empty, since X is connected and $B(x)$ is a proper subset of X for which $X/A \neq \{X\}$. Let u be any point in $Bd(B(x))$. Lemma 3.7 tells us that

$$(u, y) \in Bd(B(x) \times B(x))$$

for any $y \in B(x)$. On the other hand

$$Bd(B(x) \times B(x)) \subset \overline{B(x) \times B(x)} \subset \bar{S} = S, \quad \dots (1)$$

where

$$S = \cup \{B(x) \times B(x); x \in X\},$$

which is closed, since X/A is Hausdorff. Consequently $(u, y) \in S$ for any $y \in B(x)$. However, it is clear that for any $z \in X$ and for any

$y \in B(x)$ the point (u, y) does not belong to $B(z) \times B(z)$, so that (u, y) does not belong to S for all $y \in B(x)$. This contradicts (1). Hence there exists no $x \in X$ such that $B(x)$ is open in X , i.e., every $B(x)$ is a border set. Q.E.D.

References

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