108. *On the C^∞-Goursat Problem for 2nd Order Equations with Real Constant Coefficients*

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§ 1. **Introduction.** We consider the following Goursat problem (1.1)–(1.2).

$$\partial_t\partial_x u=\sum_{\substack{i+j+|\alpha|\leqslant 2\\ i+j\leqslant 1}} a_{ij\alpha}\partial_t^i\partial_x^j\partial_y^\alpha u, \qquad t\in R^1_+,\ x\in R^1,\ y\in R^n \tag{1.1}$$

where $a_{ij\alpha}$ are real constants

$$\begin{cases} u(0,x,y)=\varphi(x,y)\in\mathcal{E}_{xy} \\ u(t,0,y)=\psi(t,y)\in\mathcal{E}_{ty} \qquad t\geqslant 0 \\ \varphi(0,y)=\psi(0,y) \qquad \text{(compatibility condition).}\end{cases} \tag{1.2}$$

We notice that, $t=0$ and $x=0$ are characteristic hypersurfaces of the equation (1.1). We say that the Goursat problem (1.1)–(1.2) is well posed for the future in the space $\mathcal{E}$, if for any given Goursat data (1.2), there exists a unique solution $u(t,x,y)\in\mathcal{E}_{txy}$, $t\geqslant 0$, which takes the given Goursat data at $t=0$ and $x=0$.*)

Let us consider the characteristic equation (considering the lower order terms) of (1.1).

$$\lambda\xi=\sum_{\substack{1\leqslant i+j+|\alpha|\leqslant 2\\ i+j\leqslant 1}} a_{ij\alpha}\lambda^i\xi^j\eta^\alpha, \qquad \xi\in R^1,\ \eta\in R^n.$$

Then we have

$$\lambda=\sum_{j\leqslant 1,\ 1\leqslant j+|\alpha|\leqslant 2} a_{0j\alpha}\xi^j\eta^\alpha\Big/\Big(\xi-\sum_{|\alpha|\leqslant 1} a_{10\alpha}\eta^\alpha\Big). \tag{1.3}$$

Our purpose is to prove the following

Theorem 1. *The necessary and sufficient condition for the $\mathcal{E}$-wellposedness of the Goursat problem* (1.1)–(1.2) *in the neighborhood of the origin is that λ in* (1.3) *remains bounded when $|\xi|+|\eta|$ remains bounded.*

Remark 1. We can rewrite (1.1) in the following.

$$\{\partial_t-(a_1\partial_{y_1}+a_2\partial_{y_2}+\cdots+a_n\partial_{y_n}+a_0)\}\{\partial_x-(b_1\partial_{y_1}+\cdots+b_n\partial_{y_n}+b_0)\}u =\sum_{|\alpha|\leqslant 2} c_\alpha\partial_y^\alpha u. \tag{1.4}$$

The necessary and sufficient condition in the theorem 1 is equivalent to $c_\alpha=0$ for $|\alpha|\geqslant 1$.

§ 2. **Proof of Theorem 1.** At first we consider the following fairly simple equation;

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*) According to Banach's closed graph theorem, if the Goursat problem is $\mathcal{E}$-wellposed then the linear mapping $(\varphi,\psi)\to u$ is continuous from $\mathcal{E}_{xy}\times\mathcal{E}_{ty}$ into $\mathcal{E}_{txy}$.

(2.1) $$\partial_t\partial_x u = a\partial_y^2 u + b\partial_y u + cu \qquad y \in R^1.$$
We shall show

Proposition. *If the Goursat problem* (2.1)–(1.2) *is $\mathcal{E}$-wellposed in the neighborhood of the origin, then we have* $a=b=0$.

For the proof of this proposition, we use the following lemma on Bessel functions (see [1] p. 526).

Lemma. *The Bessel function*

(2.2) $$J_0(z) = \sum_{k\geqslant 0} \frac{(-z^2/4)^k}{(k\,!)^2}$$

has the following representation for large $|z|$ *inside an angle* $-\frac{1}{2}\pi+\delta$ $<\arg z<\frac{1}{2}\pi-\delta$ $(\delta>0)$,

(2.3) $$J_0(z) = \sqrt{\frac{1}{2\pi z}}\,(e^{i(z-\pi/4)} + e^{-i(z-\pi/4)}) + 0(|z|^{-2/3}).$$

The proof of Proposition. At first we suppose $a\neq 0$. We can reduce (2.1) to the following form:

(2.4) $$\partial_t\partial_x u = a\partial_y^2 u + cu.$$
Let us show that (2.4) is not $\mathcal{E}$-wellposed. We seek for the solution of (2.4) which has the form $u=v(t,x)e^{i\eta y}$ (where η is real and positive). Then we have

(2.5) $$\partial_t\partial_x v = -a\eta^2 v + cv.$$
We impose the data (2.6) on v.

(2.6) $$v(t,0)=v(0,x)=1.$$
The function

(2.7) $$v(t,x) = \sum_{k\geqslant 0} \frac{\{(-\eta^2 a + c)xt\}^k}{(k\,!)^2}$$

satisfies (2.5) and (2.6), therefore $u(t,x)$ is a solution of (2.4) and has the following Goursat data (2.8).

(2.8) $$u(0,x)=u(t,0)=e^{i\eta y}.$$
For $-ax>0$ and for sufficiently large η, we have

(2.9) $$v(t,x) \geqslant \sum_{k\geqslant 0} \frac{(\sqrt{(-\eta^2 a + c)xt})^{2k}}{(2k)\,!} = \frac{1}{2}(e^{\sqrt{(-\eta^2 a+c)xt}} + e^{-\sqrt{(-\eta^2 a+c)xt}}).$$

Hence

(2.10) $$|u(t,x,y)| > \frac{1}{2} e^{\sqrt{(-a\eta^2+c)xt}} \qquad \text{for } -ax>0 \text{ and large } \eta.$$

(2.8) and (2.10) show that the continuity from data to solution can not be held. Then "$\mathcal{E}$-wellposedness$\Rightarrow a=0$" has been proved.

Next, we suppose $a=0$, $b\neq 0$ in (2.1). In this case we can reduce (2.1) to the following

(2.11) $$\partial_t\partial_x u = b\partial_y u.$$
In the same way as the case $a\neq 0$, $u=v(t,x)e^{i\eta y}$ is a solution of (2.11),

where v is the following

$$v=\sum_{k\geq 0}\frac{(ibxt\eta)^k}{(k!)^2}. \tag{2.12}$$

From (2.2) and (2.12), we have

$$v(t,x)=J_0(2\sqrt{-bx\eta t}e^{\pi i/4}) \qquad \text{for } -bx>0. \tag{2.13}$$

In view of Lemma we have

$$|v(t,x)|>\text{constant}.\ \eta^{-1/4}e^{\sqrt{2|bx|\eta t}} \qquad \text{for large } \eta. \tag{2.14}$$

So in the same way as the case $a\neq 0$, we have "$\mathcal{E}$-wellposedness$\Rightarrow b=0$". The proof of proposition thus completes.

The proof of Theorem 1. In view of Remark 1, we consider (1.4) instead of (1.1). If $c_\alpha=0$ for $|\alpha|\geqslant 1$ in (1.4), we have

$$\{\partial_t-(a_1\partial_{y_1}+\cdots+a_n\partial_{y_n}+a_0)\}\{\partial_x-(b_1\partial_{y_1}+\cdots+b_n\partial_{y_n}+b_0)\}u \\ =cu. \tag{2.15}$$

Let us consider the following change of independent variables.

$$\begin{cases} T=t \\ X=x \\ Y_i=y_i+a_it+b_ix \qquad i=1,2,\cdots,n. \end{cases} \tag{2.16}$$

From (2.15) and (2.16), we have

$$(\partial_T-a_0)(\partial_X-b_0)u=cu. \tag{2.17}$$

Let

$$u=e^{a_0T+b_0X}\tilde{u}. \tag{2.18}$$

We have

$$\partial_T\partial_X\tilde{u}=\tilde{c}\tilde{u}. \tag{2.19}$$

Considering (2.16) and (2.18), we rewrite (1.2) in the following

$$\begin{cases} \tilde{u}|_{T=0}=\tilde{\varphi}(X,Y) \\ \tilde{u}|_{X=0}=\tilde{\psi}(T,Y). \end{cases} \tag{2.20}$$

By successive approximation we have a unique C^∞-solution of the Goursat problem (2.19)–(2.20).
To prove that the condition in Theorem 1 is necessary, changing the independent variables and unknown function, we can reduce Theorem 1 to Proposition.

Remark 2. When the hypersurface $x=0$ is not characteristic, i.e. the term ∂_x^2 appears in the right hand side of (1.1), we have some results which are analogous to Theorem 1.

§ 3. A result concerning system. Let us consider the following Goursat problem

$$\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ & & 1 \\ 0 & & 0 \end{pmatrix}\partial_t u-\sum_{k=1}^n A_k\partial_{x_k}u-Bu=0 \qquad t\in R_+^1,\ x\in R^n \tag{3.1}$$

$$\begin{cases} u_i(0,x)=u_i(x) \qquad i=1,2,\cdots,N-1 \\ u_N(t,x)|_{x_1=0}=u_N(t,x') \qquad x'=(x_2,x_3,\cdots,x_n) \end{cases} \tag{3.2}$$

where A_k and B are matrices of order N, each components are $C^\infty_{t,x}$

functions, and (N, N) component of A_1 is 1.

We consider the following characteristic polynomial;

$$(3.3)\qquad \det\left\{\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ & & 1 \\ 0 & & 0 \end{pmatrix}\lambda - \sum_{k=1}^{n} A_k(0,0)\xi_k\right\} = b_1(\xi)\lambda^{N-1} + \cdots + b_N(\xi).$$

Our result is the following.

"*Assume that, for some* ξ^0 $(\neq 0)$ *real,* $b_1(\xi^0)\neq 0$ *and the polynomial* $b_1(\xi^0)\lambda^{N-1} + \cdots + b_N(\xi^0)$ *has a non real root, then Goursat problem* (3.1)–(3.2) *is not* $\mathcal{E}$*-wellposed in any small neighborhood of the origin*".

The proof of Remark 2 and § 3 will be given in a forthcoming paper.

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Reference

[1] R. Courant and D. Hilbert: Methods of Mathematical Physics 1. Interscience publishers. New York (1953).