106. *Divisors on Meromorphic Function Fields*

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Consider the field $M(R)$ of meromorphic functions on an *open* Riemann surface R. Let

$$f(z)=\sum_{n=-\infty}^{\infty} b_n(z-a)^n$$

be the Laurent expansion of an $f\in M(R)^*=M(R)-\{0\}$ at a point $a\in R$ where we use the same notation for generic points of R and their local parameters. The divisor $\partial_f(a)$ of $f\in M(R)^*$ at $a\in R$ is defined by

$$\partial_f(a)=\inf\{n\,;\,b_n\neq 0\}.$$

We first fix an $f\in M(R)^*$ in $\partial_f(a)$ and consider it as a function of a on R. Extracting the essence of the point function $\partial_f(\cdot):R\to \boldsymbol{Z}$ (the integers) we call a mapping $\partial(\cdot):R\to\boldsymbol{Z}$ a *divisor on* R if the set $\{z\in R\,;\,\partial(z)\neq 0\}$ is isolated in R. Then we have

(I) **The Weierstrass-Florack Theorem.** *For any divisor $\partial(\cdot)$ on R there exists a unique (up to multiplications by zero free holomorphic functions) $f\in M(R)^*$ such that $\partial(\cdot)=\partial_f(\cdot)$ on R.*

We next fix a point $a\in R$ in $\partial_f(a)$ and consider it as a functional of f on $M(R)^*$. As an abstraction of the functional $\partial.(a):M(R)^*\to\boldsymbol{Z}$ we say that a mapping $\partial.:M(R)^*\to\boldsymbol{Z}$ is a *divisor on* $M(R)^*$ if the following four conditions are satisfied:

(α) $\partial_{M(R)^*}=\boldsymbol{Z}$;

(β) $\partial_{C^*}=\{0\}$;

(γ) $\partial_{f\cdot g}=\partial_f+\partial_g$;

(δ) $\partial_{f+g}\geq\min(\partial_f,\partial_g)$,

where $\boldsymbol{C}$ is the field of complex numbers and $\boldsymbol{C}^*=\boldsymbol{C}-\{0\}$. As a counter part to (I) we have

(II) **The Iss'sa Theorem.** *For any divisor $\partial.$ on $M(R)^*$ there exists a unique point $a\in R$ such that $\partial.=\partial.(a)$ on $M(R)^*$.*

The crucial part of the proof of Iss'sa [2] of the above theorem is to show that

$(*)$ $\qquad\partial_z\geq 0$ *for any divisor $\partial.$ on $M(\boldsymbol{C})^*$.*

Observe that $M(R)=\{f/g\,;\,f,g\in A(R),\ g\not\equiv 0\}$ where $A(R)$ is the ring of holomorphic functions on R. Hence

$$d_f=\inf\{|\partial_g|\,;\,g\in M(\boldsymbol{C})^*\circ f,\ \partial_g\neq 0\}$$

is an integer not less than 1 for any fixed $f\in A(R)^*=A(R)-\{0\}$ with $\partial_f\neq 0$ and any fixed divisor $\partial.$ on $M(R)^*$, and there in fact exists a

$g_0 \in M(\boldsymbol{C})^* \circ f$ with $d_f = \partial_{g_0}$. For any $g \in M(\boldsymbol{C})^* \circ f$ with $\partial_g \geq 0$, let $\partial_g = md_f + n$ where m and n are integers with $0 \leq n < d_f$. Then

$$\partial_g = \partial_{g_0^m} + \partial_{g/g_0^m} = md_f + \partial_{g/g_0^m}$$

and therefore $\partial_{g/g_0^m} = n$ with $g/g_0^m \in M(\boldsymbol{C})^* \circ f$. This implies that $n=0$, i.e. d_f divides ∂_g. The same is true for $\partial_g \leq 0$ since $\partial_{g^{-1}} = -\partial_g \geq 0$. A fortiori

$$\bar{\partial}_\varphi = \frac{1}{d_f} \partial_{\varphi \circ f}$$

is a divisor on $M(\boldsymbol{C})^*$. Once $(*)$ is established, then we deduce that $\partial_f = \partial_{z \circ f} = d_f \bar{\partial}_z \geq 0$, i.e.

$(**)$ $\partial_f \geq 0$ *for any* $f \in A(R)^*$ *and any divisor* $\partial.$ *on* $M(R)^*$.

Only this is sufficient to prove that any field isomorphism of $M(R_1)$ onto $M(R_2)$ is also a ring isomorphism of $A(R_1)$ onto $A(R_2)$, where R_1 and R_2 are open Riemann surfaces, since, by $(**)$, any $f \in M(R)^*$ belongs to $A(R)$ if and only if $\partial_f \geq 0$ for any divisor $\partial.$ on $M(R)^*$. Thus in particular $M(R_1)$ and $M(R_2)$ are field isomorphic if and only if R_1 and R_2 are direct or indirect conformally equivalent (Iss'sa [2], cf. also Alling [1]). The deduction of (II) from $(**)$ is rather elementary except for the use of (I) (see Iss'sa [2]).

The purpose of this note is, modifying the original proof of Iss'sa, to give a less algebraic elementary proof to $(*)$ avoiding the explicit use of the valuation theory such as p-adic integers so that the Iss'sa theorem can also be included in the class use textbooks of complex function theory as one of attractive applications of the Weierstrass theorem (see Heins [3, Chap. VIII]).

Proof of $(*)$. Contrary to the assertion assume that $\partial_z < 0$. We maintain that $\partial_{z-c} = \partial_z$ for every $c \in \boldsymbol{C}$. In fact, $\partial_{z-c} \geq \min(\partial_z, \partial_c) = \partial_z = \partial_{z-c+c} \geq \min(\partial_{z-c}, \partial_c) = \partial_{z-c}$ since $\partial_z < 0$. Let $m = -\partial_z + 2$ and consider a divisor $\partial(\cdot)$ on $\boldsymbol{C}$ such that $\partial(j) = m^j$ for $j \in \boldsymbol{Z}^+ = \{j \in \boldsymbol{Z}; j \geq 0\}$ and $\partial(a) = 0$ for $a \in \boldsymbol{C} - \boldsymbol{Z}^+$. By (I) there exists an $f \in A(\boldsymbol{C})^*$ such that $\partial(\cdot) = \partial_f(\cdot)$ on $\boldsymbol{C}$. Fix an arbitrary natural number n and set $f_n = f / \prod_{j=0}^{n-1} (z-j)^{m^j}$. Then

$$\partial_{f_n} = \partial_f - \sum_{j=0}^{n-1} m^j \partial_{z-j}$$

and therefore

$$(1) \qquad (m-1)\partial_{f_n} = \{(m-1)\partial_f + \partial_z\} - m^n \partial_z.$$

Consider another divisor $\bar{\partial}(\cdot)$ on $\boldsymbol{C}$ such that $\bar{\partial}(j) = m^{j-n}$ for $j \in (n + \boldsymbol{Z}^+)$ and $\bar{\partial}(a) = 0$ for $a \in \boldsymbol{C} - (n + \boldsymbol{Z}^+)$. Then once more by (I) there exists a $g_n \in A(\boldsymbol{C})^*$ such that $\bar{\partial}(\cdot) = \partial_{g_n}(\cdot)$ on $\boldsymbol{C}$. Since $f_n / g_n^{m^n}$ is in $A(\boldsymbol{C})^*$ and zero free, the simply connectedness of $\boldsymbol{C}$ assures the existence of the $m^n th$ root $F_n \in A(\boldsymbol{C})^*$ of $f_n / g_n^{m^n}$, i.e. $F_n^{m^n} = f_n / g_n^{m^n}$. Put $G_n = F_n \cdot g_n$ and observe that $G_n^{m^n} = f_n$. Replacing ∂_{f_n} in (1) by $\partial_{f_n} = m^n \partial_{G_n}$ we obtain

$$m^n(m-1)\partial_{G_n}=\{(m-1)\partial_f+\partial_z\}-m^n\partial_z.$$

This shows that $(m-1)\partial_f+\partial_z$ is divisible by m^n for every $n\in \boldsymbol{Z}^+$. A fortiori, $(m-1)\partial_f+\partial_z=0$. On replacing ∂_z by $-m+2$, we finally arrive at a contradiction

$$\partial_f=(m-2)/(m-1)\notin \boldsymbol{Z}$$

since $m>2$.

References

[1] N. Alling: The valuation theory of meromorphic function fields. Proc. Symposia in Pure Math., **11**, 8–29 (1968).

[2] H. Iss'sa: On meromorphic function fields on a Stein variety. Ann. Math., **83**, 34–46 (1966).

[3] M. Heins: Complex Function Theory. Academic Press (1968).