# 146. *A Vietoris Theorem in Shape Theory*

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**1. Introduction.** In this paper the notion of shape is understood in the sense of Mardešić [2] and our approach to shape theory (cf. [5], [6]) will be used.

Our approach enables us to define the $k$-th homotopy pro-group $\pi_k\{(X, x_0)\}$ of a pointed topological space $(X, x_0)$. The homotopy pro-groups play the central role in the Whitehead theorem in shape theory.

**Theorem 1.0** (Morita [6]). *Let $f:(X, x_0)\to(Y, y_0)$ be a shape morphism of pointed connected topological spaces. If the induced morphism $\pi_k(f):\pi_k\{(X, x_0)\}\to\pi_k\{(Y, y_0)\}$ of homotopy pro-groups is an isomorphism for $1\leqq k\leqq n$ and an epimorphism for $k=n+1$ where $n+1=\max(1+\dim X, \dim Y)<\infty$, then $f$ is a shape equivalence.*

In this paper, by using homotopy pro-groups we shall formulate a Vietoris theorem in shape theory as follows.

**Theorem 1.1.** *Let $f:(X, x_0)\to(Y, y_0)$ be a closed continuous map from a pointed metrizable space $(X, x_0)$ onto a pointed topological space $(Y, y_0)$ such that $f^{-1}(y)$ is approximatively $k$-connected for every point $y$ of $Y$ and for $0\leqq k\leqq n$. Then the induced morphism $\pi_k(f):\pi_k\{(X, x_0)\}\to\pi_k\{(Y, y_0)\}$ of homotopy pro-groups is an isomorphism for $1\leqq k\leqq n$ and an epimorphism for $k=n+1$.*

The following is a direct consequene of Theorems 1.0 and 1.1 as far as $X$ is connected or locally connected.

**Theorem 1.2.** *Let $f$ be the same as in Theorem* 1.1. *If, in addition,* $\dim X\leqq n$ *and* $\dim Y\leqq n+1$, *then $f$ is a shape equivalence.*

As is quoted in [3, p. 319], in the first version of [5] we defined the $k$-th shape group $\underline{\pi}_k(X, x_0)$ of a pointed topological space $(X, x_0)$ to be the inverse limit of $\pi_k\{(X, x_0)\}$. For metric compacta M. Moszyňska [8] proved that the shape groups are naturally isomorphic to the fundamental groups in the sense of K. Borsuk. Thus, our Theorem 1.1 extends a result for metric compacta which was announced by S. Bogaty [1] and proved by K. Kuperberg [9].

**2. Preliminaries.** Let $X$ be a metrizable space. Then there is a metric space $X_0$ which is an $ANR$ for metric spaces and contains $X$ as its closed subset. Let $f:(X, x_0)\to(Y, y_0)$ be a closed continuous map from $(X, x_0)$ onto a pointed topological space $(Y, y_0)$. Then the collection $\{f^{-1}(y)\mid y\in Y\}\cup\{\{x\}\mid x\in X_0-X\}$ of subsets of $X_0$ defines an upper semi-continuous decomposition of $X_0$ and the decomposition space $Y_0$. Then the quotient map $f_0: X_0 \to Y_0$ is a closed continuous onto map such that $f = f_0 | X$ and $f_0^{-1}(Y) = X$, and $Y_0$ is perfectly normal and paracompact.

Let $\{\mathfrak{V}_\lambda | \lambda \in \Lambda\}$ be the set of all the collections of open subsets of $Y_0$ satisfying the following conditions:

(1) $Y \subset H_\lambda$, where $H_\lambda = \cup\{V | V \in \mathfrak{V}_\lambda\}$,

(2) $\mathfrak{V}_\lambda$ is locally finite in $Y_0$,

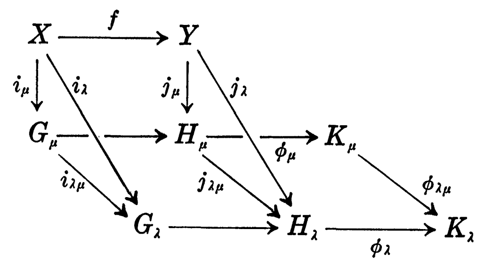
(3) the correspondence $V \to V \cap Y$ for $V \in \mathfrak{V}_\lambda$ defines an isomorphism from $N(\mathfrak{V}_\lambda)$ to $N(\mathfrak{V}_\lambda \cap Y)$,

(4) only one member of $\mathfrak{V}_\lambda$ contains $y_0$.

Here $N$ means the operation of taking the nerve of a cover. For $\lambda$, $\mu \in \Lambda$ let us define $\lambda \leq \mu$ by requiring that $\mathfrak{V}_\mu$ is a refinement of $\mathfrak{V}_\lambda$. Thus $\lambda \leq \mu$ implies $H_\mu \subset H_\lambda$. Let $K_\lambda$ be $N(\mathfrak{V}_\lambda)$ and $k_{0\lambda}$ the vertex of $K_\lambda$ corresponding to the member of $\mathfrak{V}_\lambda$ containing $y_0$ (cf. (4)), and let us put $G_\lambda = f_0^{-1}(H_\lambda)$.

Then by [4, Lemma 1][1] the set $\{\mathfrak{V}_\lambda \cap Y | \lambda \in \Lambda\}$ of covers of $Y$ is cofinal in the set of all locally finite normal open covers of $Y$ with respect to the order by refinement. On the other hand, since $f_0$ is a closed map, $\{G_\lambda | \lambda \in \Lambda\}$ is cofinal in the set of all open neighborhoods of $X$ in $X_0$ with respect to the order by inclusion.

Therefore, the inverse system $\{(G_\lambda, x_0), i_{\lambda\mu}, \Lambda\}$ with the inclusion maps $i_{\lambda\mu}$ as bonding maps induces an inverse system in $\mathfrak{W}_0$ which is associated with $(X, x_0)$ (cf. [5, Theorem 1.4]) and $\{(K_\lambda, k_{0\lambda}), [\phi_{\lambda\mu}], \Lambda\}$ is an inverse system in $\mathfrak{W}_0$ which is associated with $(Y, y_0)$ (cf. [5, Theorem 1.3]), where $\mathfrak{W}_0$ is the homotopy category of topological spaces having the homotopy type of a $CW$ complex and $\phi_{\lambda\mu}: (K_\mu, k_{0\mu}) \to (K_\lambda, k_{0\lambda})$ for $\lambda$, $\mu \in \Lambda$ with $\lambda \leq \mu$ are canonical projections. Let $\phi_\lambda: (H_\lambda, y_0) \to (K_\lambda, k_{0\lambda})$ be a canonical map for $\lambda \in \Lambda$ such that $\phi_\lambda^{-1}(\mathrm{St}(v; K_\lambda)) = V$. Then we have the homotopy commutative diagram:




where the description of base-points is omitted and $i_\lambda, i_\mu, i_{\lambda\mu}, j_\lambda, j_\mu$ and $j_{\lambda\mu}$ are all inclusion maps.

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1) This lemma remains valid even in case $X$ is a countably paracompact, collectionwise normal space.

Let $f_\lambda:(G_\lambda, x_0)\to(K_\lambda, k_{0\lambda})$ be a map defined by $f_\lambda(x)=\phi_\lambda f_0(x)$ for $x\in G_\lambda$. Then $\{1, f_\lambda, \Lambda\}$ is a special system map from the inverse system $\{(G_\lambda, x_0), [i_{\lambda\mu}], \Lambda\}$ to the inverse system $\{(K_\lambda, k_{0\lambda}), [\phi_{\lambda\mu}], \Lambda\}$ which represents a shape morphism from $(X, x_0)$ to $(Y, y_0)$ induced by $f$.

**3. Proof of Theorem 1.1.** Let $f:(X, x_0)\to(Y, y_0)$ be the same as in § 2. Moreover, assume that $f^{-1}(y)$ is approximatively $k$-connected for each point $y$ of $Y$ and for $0\leqq k\leqq n$. Let us keep the notation in § 2. We shall say that a subset $A$ of a space $B$ is $\pi_k$-trivially embedded in $B$ if every continuous map from a $k$-sphere $S^k$ to $A$ is null homotopic in $B$. Thus, a subset $C$ of $X_0$ is approximatively $k$-connected iff each open neighborhood $U$ of $C$ embedds an open neighborhood $V$ of $C$ $\pi_k$-trivially. For collections $\mathfrak{U}$ and $\mathfrak{B}$ of subsets of $X_0$, we shall say that $\mathfrak{U}$ refines $\mathfrak{B}$ $\pi_k$-trivially if each member of $\mathfrak{U}$ is $\pi_k$-trivially embedded in some member of $\mathfrak{B}$.

A partial realization of a polyhedron (=a simplicial complex with the weak topology) $P$ in $f_0^{-1}(\mathfrak{B}_\mu)$ is a continuous map $g: Q\to X_0$ of some subpolyhedron $Q\subset P$ containing the zero-skeleton $P^0$ of $P$, such that $g(Q\cap\sigma)$ is contained in some $f_0^{-1}(V)$ with $V\in\mathfrak{B}_\mu$ for each closed simplex $\sigma$ of $P$. The realization of $P$ is called full if $Q=P$. The following lemma is easy to see.

**Lemma 3.1.** *Let $\{\lambda_0, \lambda_1, \cdots, \lambda_{n+1}\}$ be a sequence of elements of $\Lambda$ such that $f_0^{-1}(\mathrm{St}(\mathfrak{B}_{\lambda_k}))$ refines $f_0^{-1}(\mathfrak{B}_{\lambda_{k+1}})$ $\pi_k$-trivially for $0\leqq k\leqq n$, where $\mathrm{St}(\mathfrak{B}_{\lambda_k})=\{\mathrm{St}(V, \mathfrak{B}_{\lambda_k})\mid V\in\mathfrak{B}_{\lambda_k}\}$. Then any partial realization of a polyhedron $P$, with $\dim P\leqq n+1$, in $f_0^{-1}(\mathrm{St}(\mathfrak{B}_{\lambda_0}))$ can be extended to a full realization of $P$ in $f_0^{-1}(\mathfrak{B}_{\lambda_{n+1}})$.*

We write $\lambda\prec\mu$ is case there is a sequence $\{\lambda_0, \lambda_1, \cdots, \lambda_{n+1}\}$ in $\Lambda$ satisfying the condition of Lemma 3.1 such that $\mathfrak{B}_{\lambda_{n+1}}$ is a star-refinement of $\mathfrak{B}_\lambda$ and $\lambda_0\leq\mu$.

**Lemma 3.2.** *For any $\lambda\in\Lambda$ there is some $\mu\in\Lambda$ with $\lambda\prec\mu$.*

**Lemma 3.3.** *For any $\lambda, \mu\in\Lambda$ with $\lambda\prec\mu$ there is a continuous map $g_{\lambda\mu}:(K_\mu^{n+1}, k_{0\mu})\to(G_\lambda, x_0)$ such that $f_\lambda g_{\lambda\mu}\simeq\phi_{\lambda\mu}\mid K_\mu^{n+1}:(K_\mu^{n+1}, k_{0\mu})\to(K_\lambda, k_{0\lambda})$.*

**Proof.** To each vertex $v_{\mu,\beta}$ of $K_\mu$ let us assign a point $g_0(v_{\mu,\beta})\in f_0^{-1}(V_{\mu,\beta})$ and define a map $g_0: K_\mu^0\to X_0$. Here we denote by $v_{\mu,\beta}$ the vertex of $K_\mu$ corresponding to the member $V_{\mu,\beta}$ of $\mathfrak{B}_\mu$. Let $v_{\mu,\beta_i}, i=0, 1, \cdots, r$, be vertices of a simplex $\sigma$ of $K_\mu$. Then $g_0(\sigma\cap K_\mu^0)\subset\cup\{f_0^{-1}(V_{\mu,\beta_i})\mid 0\leq i\leq r\}\subset\mathrm{St}(f_0^{-1}(V_{\mu,\beta_0}), f_0^{-1}(\mathfrak{B}_\mu))\subset\mathrm{St}(f_0^{-1}(V_{\lambda_0,\alpha_0}), f_0^{-1}(\mathfrak{B}_{\lambda_0}))$. Hence $g_0$ is a partial realization of $K_\mu$ in $f_0^{-1}(\mathrm{St}(\mathfrak{B}_{\lambda_0}))$. By Lemma 3.1 $g_0$ is extended to a partial realization $g_{n+1}: K_\mu^{n+1}\to X_0$ in $f_0^{-1}(\mathfrak{B}_{\lambda_{n+1}})$.

Let $v_{\mu,\beta_i}, i=0, 1, \cdots, r$, be vertices of a simplex $\sigma^r$ in $K_\mu$ with $r\leqq n+1$. Suppose that

(6) $f_0^{-1}(V_{\mu,\beta_0})\subset f_0^{-1}(V_{\lambda_{n+1},\alpha_0})$ with $V_{\lambda_{n+1},\alpha_0}\in\mathfrak{B}_{\lambda_{n+1}}$,

(7) $g_{n+1}(\sigma^r)\subset f_0^{-1}(V_{\lambda_{n+1},\alpha})$ with $V_{\lambda_{n+1},\alpha}\in\mathfrak{B}_{\lambda_{n+1}}$.

Since $g_{n+1}(v_{\mu,\beta_0})\subset f_0^{-1}(V_{\lambda_{n+1},\alpha_0})$, we have $g_{n+1}(\sigma^r)\subset \mathrm{St}\,(f_0^{-1}(V_{\lambda_{n+1},\alpha_0}),$ $f_0^{-1}(\mathfrak{V}_{\lambda_{n+1}}))$. Hence $g_{n+1}(\mathrm{St}(v_{\mu,\beta_0}\,;K_\mu^{n+1}))\subset \mathrm{St}\,(f_0^{-1}(V_{\lambda_{n+1},\alpha_0}),f_0^{-1}(\mathfrak{V}_{\lambda_{n+1}}))$.

Suppose that $\mathrm{St}\,(f_0^{-1}(V_{\lambda_{n+1},\alpha_0}),f_0^{-1}(\mathfrak{V}_{\lambda_{n+1}})\subset f_0^{-1}(V_{\lambda,r})$ with $V_{\lambda,r}\in\mathfrak{V}_\lambda$. Then we have

$$f_\lambda g_{n+1}(\mathrm{St}\,(v_{\mu,\beta_0}\,;K_\mu^{n+1}))\subset \mathrm{St}\,(v_{\lambda,r}\,;K_\lambda),$$

and $V_{\mu,\beta_0}\subset V_{\lambda,r}$. Thus, if we put $g_{\lambda\mu}=g_{n+1}$ and define a map $\phi'_{\lambda\mu}:K_\mu\to K_\lambda$ by $\phi'_{\lambda\mu}(v_{\mu,\beta_0})=v_{\lambda,r}$, then $\phi'_{\lambda\mu}$ is a canonical projection and $\phi'_{\lambda\mu}|K_\mu^{n+1}:K_\mu^{n+1}\to K_\lambda$ is a simplicial approximation of $f_\lambda g_{\lambda\mu}$. Hence we have $f_\lambda g_{\lambda\mu}\simeq\phi_{\lambda\mu}|K_\mu^{n+1}:(K_\mu^{n+1},k_{0\mu})\to(K_\lambda,k_{0\lambda})$.

**Lemma 3.4.** *Suppose that $\lambda\prec\mu$. Then any continuous map $\xi$ from $(S^k,s_0)$ to $(G_\mu,x_0)$ such that $f_\mu\xi:(S^k,s_0)\to(K_\mu,k_{0\mu})$ is null homotopic is null homotopic in $(G_\lambda,x_0)$ for $k\leqq n$.*

**Proof.** Suppose that $S^k$ is the boundary of $I^{k+1}$ where $I=[0,1]$. Then the map $f_\mu\xi$ is extended to a continuous map $\chi:I^{k+1}\to K_\mu$. Let $P$ be a simplicial subdivision of $I^{k+1}$ such that for a closed simplex $\sigma$ in $P$ $\chi(\sigma)$ is contained in $\mathrm{St}(v\,;K_\mu)$ with some vertex $v$ of $K_\mu$ and such that a subcomplex $Q$ of $P$ is a subdivision of $S^k$. For $p\in Q$ let us put $\psi(p)=\xi(p)$ and for a vertex $w$ of $P-Q$ let $\psi(w)$ be a point of $f_0^{-1}(V_{\mu,\beta})$ if $\chi(w)\in\mathrm{St}\,(v_{\mu,\beta}\,;K_\mu)$. If $\tau$ is a closed simplex of $Q$ and $\chi(\tau)=f_\mu\xi(\tau)\subset\mathrm{St}(v_{\mu,r}\,;K_\mu)$, then $\psi(\tau)=\xi(\tau)\in f_\mu^{-1}(\mathrm{St}\,(v_{\mu,r}\,;K_\mu))\subset f_0^{-1}(V_{\mu,r})$.

Let $\sigma$ be a closed simplex of $P$ such that $\tau=\sigma\cap Q$ and $w_j$, $0\leqq j\leqq m$ are vertices of $\sigma$ lying not in $Q$. Then there is a vertex $v_{\mu,\alpha}$ of $K_\mu$ such that $\chi(\sigma)\subset\mathrm{St}\,(v_{\mu,\alpha}\,;K_\mu)$. Suppose that $\chi(w_i)\subset\mathrm{St}\,(v_{\mu,\alpha_i}\,;K_\mu)$ for $0\leqq i\leqq m$. Then $V_{\mu,\alpha_i}\cap V_{\mu,\alpha}\neq\emptyset$, $V_{\mu,r}\cap V_{\mu,\alpha}\neq\emptyset$. Hence $\psi(\sigma\cap(Q\cup P^0))\subset\mathrm{St}\,(f_0^{-1}(V_{\mu,\alpha}),$ $f_0^{-1}(\mathfrak{V}_\mu))$.

Thus, $\psi$ is a partial realization of $P$ in $f_0^{-1}(\mathrm{St}\,(\mathfrak{V}_\mu))$. Since $\dim P\leqq n+1$, by Lemmas 3.1 and 3.2 $\psi$ is extended to a full realization of $P$ in $f_0^{-1}(\mathfrak{V}_\lambda)$. Hence $i_{\lambda\mu}\xi:(S^k,s_0)\to(G_\lambda,x_0)$ is null homotopic.

Now, we are in a position to prove Theorem 1.1.

**Proof of Theorem 1.1.** Let $\lambda\prec\mu$. Then by Lemmas 3.2 and 3.4 we have $\mathrm{Im}\,[\pi_k(\phi_{\lambda\mu})]\subset\mathrm{Im}\,[\pi_k(f_\lambda)]$ for $0\leqq k\leqq n+1$, and $\pi_k(i_{\lambda\mu})\,\mathrm{Ker}\,[\pi_k(f_\mu)]=0$ for $0\leqq k\leqq n$. Therefore, by [6, Theorem 1.2] $\pi_k(f):\pi_k\{(X,x_0)\}\to\pi_k\{(Y,y_0)\}$ is a monomorphism for $1\leqq k\leqq n$ and an epimorphism for $1\leqq k\leqq n+1$. This completes the proof of Theorem 1.1 by [6, Theorem 1.3] or [10, Theorem 2].

## 4. Proof of Theorem 1.2.

In addition to the assumption in §3 we shall assume here that $\dim Y\leqq n+1$.

**Lemma 4.1.** *Let $\lambda\prec\mu$ and let $\xi:(P,p_0)\to(G_\mu,x_0)$ be a continuous map, where $(P,p_0)$ is a pointed polyhedron. Then there is a simplicial subdivision $P_1$ of $P$ such that for each closed simplex $\sigma$ of $P_1$ there is $V\in\mathfrak{V}_\lambda$ with $f_0^{-1}(V)\supset\xi(\sigma)\cup g_{\lambda\mu}f_\mu\xi(\sigma)$.*

**Proof.** Let $P_1$ be a simplicial subdivision of $P$ such that for each closed simplex $\sigma$ of $P_1$ there is $V \in \mathfrak{V}_\mu$ with $\xi(\sigma) \subset f_0^{-1}(V)$.

Suppose that $\xi(\sigma) \subset f_0^{-1}(V_{\mu,\beta_0})$ for a closed simplex $\sigma$ of $P_1$ and for $V_{\mu,\beta_0} \in \mathfrak{V}_\mu$. Then we have $f_\mu \xi(\sigma) \subset \mathrm{St}\,(v_{\mu,\beta_0}; K_\mu)$. Since $\dim Y \leqq n+1$, we can assume that $K_\mu^{n+1} = K_\mu$. Hence, by the proof of Lemma 3.3, we have $g_{\lambda\mu} f_\mu \xi(\sigma) \subset \mathrm{St}\,(f_0^{-1}(V_{\lambda_{n+1},\alpha_0}), f_0^{-1}(\mathfrak{V}_{\lambda_{n+1}}))$, where $V_{\mu,\beta_0} \subset V_{\lambda_{n+1},\alpha_0} \in \mathfrak{V}_{\lambda_{n+1}}$, and consequently $\xi(\sigma) \cup g_{\lambda\mu} f_\mu \xi(\sigma) \subset \mathrm{St}\,(f_0^{-1}(V_{\lambda_{n+1},\alpha_0}), f_0^{-1}(\mathfrak{V}_{\lambda_{n+1}}))$. This proves Lemma 4.1.

As a direct consequence of Lemmas 3.1 and 4.1 we have

**Lemma 4.2.** *Let $\lambda \prec \mu, \mu \prec v$ and let $\xi : (P, p_0) \to (G_\nu, x_0)$ be a continuous map, where $P$ is a polyhedron of dimension $\leqq n$. Then $i_{\lambda\nu}\xi \simeq i_{\lambda\mu} g_{\mu\nu} f_\nu \xi : (P, p_0) \to (G_\lambda, x_0)$.*

We are now able to prove Theorem 1.2.

**Proof of Theorem 1.2.** Assume that $\dim X \leqq n$. Let $\lambda \prec \mu, \mu \prec \nu$. Since the Čech system of $(X, x_0)$ (cf. [5]) and $\{(G_\lambda, x_0), i_{\lambda\mu}, \Lambda\}$ are isomorphic in pro $(\mathfrak{W}_0)$, there is $\kappa \in \Lambda$ with $\nu \leq \kappa$ such that for some polyhedron $P$ of dimension $\leqq n$ there are continuous maps $\xi : (P, p_0) \to (G_\nu, x_0), \eta : (G_\kappa, x_0) \to (P, p_0)$ with $i_{\nu\kappa} \simeq \xi\eta$. Hence by Lemma 4.2 we have $i_{\lambda\kappa} \simeq i_{\lambda\mu} g_{\mu\nu} f_\nu i_{\nu\kappa}$. On the other hand, by Lemma 3.3 we have $f_\mu g_{\mu\nu} \simeq \phi_{\mu\nu}$. Hence $i_{\lambda\kappa} \simeq \psi_{\lambda\kappa} f_\kappa$, $\phi_{\lambda\kappa} \simeq f_\lambda \psi_{\lambda\kappa}$, where $\psi_{\lambda\kappa} = i_{\lambda\mu} g_{\mu\nu} \phi_{\nu\kappa}$. Therefore, by [6, Theorem 1.1], $f$ is a shape equivalence.

The following is also a direct consequence of Lemmas 3.3 and 4.2 (cf. [7, Theorem 4.3].

**Theorem 4.3.** *Let $f$ be the same as in Theorem 1.1. Then for a pointed space $(P, p_0)$ of dimension $\leqq n$ the map $f_\# : \mathfrak{S}_0[P, X] \to \mathfrak{S}_0[P, Y]$ induced by $f$ is bijective, where $\mathfrak{S}_0[P, X]$ means the set of shape morphisms from $(P, p_0)$ to $(X, x_0)$.*

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