170. *On Continuation of Regular Solutions of Linear Partial Differential Equations with Real Analytic Coefficients*

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In [2], [3] and [4] we studied on continuation of real analytic solutions of partial differential equations with constant coefficients by way of the Fourier transform and the Fundamental Principle. Recently, in [5] we have given a new method for the problem by way of the boundary value theory. This method allows us to treat the equations with real analytic coefficients as is seen below.

§ 1. Estimate of the singular spectrum of the boundary values of real analytic solutions. Let $U\subset \boldsymbol{R}^n$ be an open neighborhood of the origin. Put $U^{\pm}=U\cap\{\pm x_1>0\}$. We denote the partition of the variables by $x=(x_1, x')=(x_1, x'', x_n)$. We employ similar notation for the dual variables ξ. Let $p(x, D)$ be an m-th order linear partial differential operator with real analytic coefficients. Assume that $x_1=0$ is non-characteristic. Then for a solution u of $p(x, D)u=0$ in U^+, we can define the boundary values $b_j^+(u)$, $j=0, 1, \cdots, m-1$, of u to $x_1=0$. Intuitively this is equal to $b_j(x, D)u|_{x_1=+0}$, where b_j are the j-th order component of the boundary system dual to $\{(-\partial/\partial x_1)^j\}_{j=0}^{m-1}$. It is characterized by the identity:

$$p(x, D)[u]=\sum_{j=0}^{m-1} b_j^+(u)\delta^{(m-j-1)}(x_1),$$

where $[u]\in\mathscr{B}(U)$ is the canonical extension of u satisfying $\operatorname{supp}[u]\subset\{x_1\geq 0\}$ (see [7]).

Theorem 1. *Let $p(x, D)$ be as above. Assume further that the principal part $p_m(x, D)$ is of principal type and has real coefficients. Let $V_{(\pm1,0,\dots,0)}(p)$ be the set of points $(x;\sqrt{-1}\xi'\infty)\in \boldsymbol{R}^n\times\sqrt{-1}\boldsymbol{S}_\infty^{n-2}$ such that the algebraic equation $p_m(x, \zeta_1, \xi')=0$ in ζ_1 has a non-real root or a real multiple root. Then for every real analytic solution u of $p(x, D)u=0$ in U^+ we have*

$$S.S.\, b_j^+(u)\subset\overline{V_{(\pm1,0,\dots,0)}(p)}|_{x_1=0}.$$

Proof. Let V be an open set in $x_1=0$ such that $V\subset\subset U\cap\{x_1=0\}$. We can assume that u is a real analytic solution on $\{0<x_1\leq\varepsilon\}\times V$. Let $[[u]]$ be a modification of $[u]$ such that $\operatorname{supp}[[u]]\subset\boldsymbol{R}\times\overline{V}$ and $[[u]]=[u]$ in $\{x_1\leq\varepsilon\}\times V$. For $x_1>0$ we can emply the characteristic function $\chi(x')$ of V and put $[[u]]=\chi(x')u$. Then we have

$$p(x, D)[[u]]+v=\sum_{j=0}^{m-1} f_j(x')\delta^{(m-j-1)}(x_1),$$

where $\operatorname{supp} v \subset \boldsymbol{R} \times \partial V$, $\operatorname{supp} f_j \subset \overline{V}$ and $f_j=b_j^+(u)$ in V. We can assume that $[[u]]$ and v contain x_1 as a real analytic parameter on $x_1>0$. Put

$$W(x', \omega')=\frac{(n-2)!\,((1-\sqrt{-1}x'\omega')^{n-2}-(1-\sqrt{-1}x'\omega')^{n-3}(x'^2-(x'\omega')^2))}{(-2\pi\sqrt{-1})^{n-1}(x'\omega'+\sqrt{-1}(x'^2-(x'\omega')^2))^{n-1}}.$$

Then $S.S.\, W(x', \omega')=(0; \sqrt{-1}\omega'\infty)$ for each fixed ω'. We have $\delta(x') =\int_{|\omega'|=1} W(x', \omega')d\omega'$ ([8], Chapter III, formula (2.1.1)). Thus, if $(0, y'; \sqrt{-1}\omega'\infty) \notin \overline{V_{(\pm 1, 0, \cdots, 0)}(p)}$, we can apply [6], Theorem 2.7 and find the solution $w_k(x, y', \omega')$, $k=0, \cdots, m-1$, of the following Cauchy problem for "I-hyperbolic equation":

$$(1)\quad \begin{cases} {}^t p(x, D)w_k=0, \\ (-\partial/\partial x_1)^j w_k|_{x_1=0}=\delta_{j, m-k-1}W(y'-x', \omega'), \qquad j=0, \cdots, m-1. \end{cases}$$

We have

$$(2)\quad \begin{aligned} &\int \{p(x, D)[[u]]+v\}w_k(x, y', \omega')Y(\varepsilon-x_1)dx \\ &\quad =\int \sum_{j=0}^{m-1} f_j(x')\delta^{(m-j-1)}(x_1)w_k(x, y', \omega')Y(\varepsilon-x_1)dx \\ &\quad =\int f_k(x')W(y'-x', \omega')dx'. \end{aligned}$$

On the other hand, by integration by parts this is equal to

$$(3)\quad \begin{aligned} &\int [[u]]\,{}^t p(x, D)\{w_k(x, y', \omega')Y(\varepsilon-x_1)\}dx+\int v w_k(x, y', \omega')Y(\varepsilon-x_1)dx \\ &\quad =\int \sum_{j=0}^{m-1} b_j(x, D)[[u]]|_{x_1=\varepsilon}(-\partial/\partial x_1)^{m-j-1}w_k(x, y', \omega')|_{x_1=\varepsilon}dx' \\ &\qquad +\int v w_k(x, y', \omega')Y(\varepsilon-x_1)dx. \end{aligned}$$

Assume that ε is small enough. Note that the bicharacteristic strip starting from $(0, y'; 0, \sqrt{-1}\omega'\infty)$ has a spatial trace transversal to $x_1=0$, because $\zeta_1=0$ is not a multiple root of the equation $p_m(0, y', \zeta_1, \omega')=0$ by the assumption. Thus from the proof of the above quoted theorem, we see that $S.S.\, w_k(x, y', \omega')$ is contained in the $c(\varepsilon)$-neighborhood of $\{(x, y'; \sqrt{-1}\xi\infty, \sqrt{-1}\eta'\infty); x'=y', \xi'=-\eta'=\omega'\}$, where $c(\varepsilon)\to 0$ if $\varepsilon\to 0$. Since $\operatorname{supp} v \subset \boldsymbol{R}\times\partial V$ and $b_j(x, D)[[u]]|_{x_1=\varepsilon}$ are real analytic in V, we conclude that the above function of y' is real analytic on a neighborhood of the point under consideration, depending analytically on ω'. Thus we have shown $(y'; \sqrt{-1}\omega'\infty) \notin S.S.\, f_k(x')$. q.e.d.

Remark. A more precise consideration shows that the above estimate can be sharpened about a half. For the case of constant coefficients see [5], Theorem 2.1.

§ 2. Continuation of real analytic solutions.

Theorem 2. *Let $p(x, D)$ be as in Theorem 1. Let K be a set contained in $x_1=0$, $\varphi(x')\geq 0$, where $\varphi(x')$ is a real analytic function* *of x' satisfying $\varphi(0)=0, d\varphi(0)\neq 0$. Assume that $(0;\sqrt{-1}d\varphi(0)\infty)$ $\notin \overline{V_{B(\pm 1,0,\cdots,0)}(p)}$. Then every real analytic solution u of $p(x,D)u=0$ defined on a neighborhood of the origin except on K can be continued as a hyperfunction solution to K in a smaller neighborhood.*

Proof. Consider the difference $b_j(u)=b_j^+(u)-b_j^-(u)$ of the boundary values from both sides of $x_1=0$. Clearly supp $b_j(u)$ is contained in K. By the assumption we can apply Theorem 1 and conclude that $S.S.\,b_j(u)$ do not contain $(0;\sqrt{-1}d\varphi(0)\infty)$. Thus by the Holmgren type theorem ([8], Chapter III, Proposition 2.1.3), we conclude that $b_j(u)=0$ on a neighborhood of the origin. Due to [7], Theorem 4 this implies that u can be continued as a hyperfunction solution to K in that neighborhood.

q.e.d.

Note that if in addition every bicharacteristic curve flows out of K on any small neighborhood of the origin, then by [6], Theorem 3.3′ the propagation of regularity holds and we conclude that u is in fact continued real analytically to K in a smaller neighborhood. For example, so is the case when K is an isolated point.

§3. Continuation of infinitely differentiable solutions. The following theorem is the extension of the result of Grušin [1] who treated the case of constant coefficients.

Theorem 3. *Let $p(x,D)$ be as in Theorem 1. Assume that $\overline{V_{(\pm 1,0,\cdots,0)}(p)}$ does not contain the whole fibre $\{0\}\times\sqrt{-1}\boldsymbol{S}_\infty^{n-2}$. Then, every infinitely differentiable solution u of $p(x,D)u=0$ defined on a neighborhood of the origin except the origin itself, can be continued to the origin as an infinitely differentiable solution.*

Proof. We go back to the proof of Theorem 1. This time we can cut off u such that in (2)–(3) $[[u]]$ is infinitely differentiable near $\boldsymbol{R}\times\partial V$ and v is infinitely differentiable everywhere. The proof of [6], Theorem 2.7 shows that the solution $w_k(x,y',\omega')$ of the Cauchy problem (1) is a distribution. Let Ω be an open set in $\boldsymbol{S}^{n-2}$ satisfying $\{0\}\times\sqrt{-1}\Omega\infty\subset\subset \boldsymbol{R}^n\times\sqrt{-1}\boldsymbol{S}_\infty^{n-2}\setminus\overline{V_{(\pm 1,0,\cdots,0)}(p)}$ and $\Omega^a=\Omega$, where a denotes the antipodal mapping. Thus, for $b_j(u)=b_j^+(u)-b_j^-(u)$ we conclude that supp $b_j(u)$ is concentrated at the origin and $b_j(u)*W(x',\omega')$ is infinitely differentiable for those fixed $\omega'\in\Omega$. Obviously we can get the same conclusion when we employ $\overline{W}(x',\omega')=\exp(-x'^2)W(x',\omega')$ instead of $W(x',\omega')$ in the initial conditions. Since $\delta(x')=\exp(-x'^2)\delta(x')=\int\overline{W}(x',\omega')d\omega'$, we have

$$b_j(u)=\int_\Omega b_j(u)*\overline{W}(x',\omega')d\omega'+\int_{C\Omega} b_j(u)*\overline{W}(x',\omega')d\omega'. \tag{4}$$

The Fourier image $\widetilde{b_j(u)}$ is an entire function of infra-exponential growth. By the above consideration, the first term of (4) is infinitely differentiable. Hence its Fourier image decreases on the real space faster than $(1+|\xi'|)^{-N}$ for any $N>0$. On the other hand, an elementary consideration employing the deformation of the path of integration shows that the Fourier image of the second term rapidly decreases for every $\xi'\in\Omega$ (c.f. [5], Lemma 2.3). Thus, for each fixed ξ', the entire function $b_j(u)(z\xi')$ of one variable z is of infra-exponential growth and decreases on the real axis faster than $(1+|z|)^{-N}$ for any $N>0$. The Phragmén-Lindelöf theorem and the Liouville theorem shows that it vanishes identically. Thus $b_j(u)=0$ and by [7], Theorem 4, u can be continued as a hyperfunction solution to the origin.

Finally we prove the regularity. Let $E(x,y)$ be one of the fundamental solution of ${}^tp(x,D)E=\delta(x-y)$. Then construction of [6] gives it as a distribution. Let $[u]$ be a function obtained from u by cutting off the support in an infinitely differentiable way outside a neighborhood of the origin. We have

$$\begin{aligned}[u](y)&=\int\{{}^tp(x,D)E(x,y)\}[u](x)dx\\&=\int E(x,y)p(x,D)[u](x)dx,\end{aligned}$$

in the sense of hyperfunctions. Because $p(x,D)[u]$ vanishes on a neighborhood of the origin, it is infinitely differentiable, hence so is the last term. q.e.d.

Note that the above proof can be applied to the classical solution of class C^m, thus producing the extension of u as a distribution solution.

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