# 168. *The Isometry Groups of Compact Manifolds with Non-positive Curvature*[*)]

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Let $M$ be an $n$-dimensional compact connected Riemannian manifold with negative Ricci curvature. Then a classical theorem of Bochner says that there exist no non-trivial Killing vector fields on $M$. And hence the order of the isometry group $I(M)$ of $M$ is finite. Relating to this theorem, T. Frankel obtained the following:

Let $M$ be a compact Riemannian manifold with non-positive sectional curvature $K_\sigma \leqq 0$ and with negative Ricci curvature. If $f: M \to M$ is an isometry which is continuously homotopic to the identity map, then $f$ is the identity, see [1]. This result was extended by H.B. Lawson and S. T. Yau in a more general situation. That is

**Theorem** ([5; Theorem 4, p. 225]). *Let $M$ be a compact Riemannian manifold with non-positive sectional curvature and Ricci curvature negative at some point of $M$. If $f: M \to M$ is an isometry continuously homotopic to the identity, then $f$ is the identity.*

As a corollary of this theorem, we easily have

**Lemma 1.** *Let $M$ be a manifold as in the theorem of Lawson and Yau. If $f: M \to M$ is an isometry such that $d(p, f(p)) < d(p, C(p))$ for all point $p \in M$, then $f$ is the identity.*

Here $d$ is the distance function of $M$ induced from the Riemannian metric and $C(p)$ the cut locus of $p$ in $M$.

Now, for such manifolds as in the theorem of Bochner or Lawson and Yau, it is natural to ask whether we can estimate the order of the isometry group $I(M)$ by using the geometrical terms of $M$, for example, the diameter, the injectivity radius, the sectional curvature and so on. To this problem, H. C. Im Hof gave an estimation of order of $I(M)$ for a manifold with the sectional curvature $K_\sigma$ satisfying $-b^2 \leqq K_\sigma \leqq -a^2 < 0$, $0 < a \leqq b$. In his argument, the assumption that $M$ is of negative curvature is essential.

In this note, we will give an estimation of the order of $I(M)$ for manifolds as in the theorem of Lawson and Yau in a different way from the one in H. C. Im Hof's theorem. The author thanks Prof. T. Otsuki for his kind advices.

Let $M$ be a compact Riemannian manifold. For a point $p \in M$,

*) Dedicated to Prof. S. Kashiwabara on his 60th birthday.

$T_p(M)$ denotes the tangent space of $M$ at $p$ and $\exp_p: T_p(M) \to M$ the exponential mapping. Every geodesic is parametrized with respect to the arc-length. Put $\delta := \min \{d(p, C(p)) : p \in M\}$. $\delta$ is called the injectivity radius of $M$. For a positive number $r$ and a point $p \in M$, $B_r(p)$ denotes the open metric ball in $M$ with radius $r$ centerd at $p$. By the compactness of $M$, we can choose a finite number of points $p_1, \cdots, p_k \in M$ such that

$$(*) \qquad \bigcup_{i=1}^{k} B_{\delta/4}(p_i) = M.$$

Of course $k$ depend on $\delta$ and other geometrical structure of $M$. In this situation, we have

**Proposition 1.** *Let $M$ be a compact Riemannian manifold with non-positive sectional curvature and Ricci curvature negative at some point of $M$. Then*

$$\textit{the order of } I(M) \leqq k^k.$$

**Proof.** We define a mapping $\Phi: I(M) \to \{$all the $k$-sequences $(i_1, i_2, \cdots, i_k)$, $1 \leqq i_1, i_2, \cdots, i_k \leqq k\}$ as follows: for $f \in I(M)$

$$\Phi(f) = (j_1, j_2, \cdots, j_k)$$

where for each $i=1, \cdots, k$, the number $j_i$ is the smallest of $j$ such that $f(p_i) \in B_{\delta/4}(p_j)$. $\Phi$ is well defined, because by means of $(*)$. We will show that $\Phi$ is injective. Assume $\Phi(f) = \Phi(g) = (j_1, j_2, \cdots, j_k)$ for $f, g \in I(M)$. Then we have

$$d(f(p_i), p_{j_i}) < \frac{\delta}{4} \quad \text{and} \quad d(g(p_i), p_{j_i}) < \frac{\delta}{4}, \quad i=1, \cdots, k.$$

Hence, putting $h := f^{-1} \circ g$, for each $i=1, \cdots, k$, we have

$$\begin{aligned} d(p_i, h(p_i)) &= d(p_i, f^{-1} \circ g(p_i)) = d(f(p_i), g(p_i)) \\ &\leqq d(f(p_i), p_{j_i}) + d(p_{j_i}, g(p_i)) < \frac{\delta}{2}. \end{aligned}$$

Now for each point $p \in M$, we can find a number $i_0$ such that $p \in B_{\delta/4}(p_{i_0})$ by $(*)$. So, from the above fact, we have

$$\begin{aligned} d(p, h(p)) &\leqq d(p, p_{i_0}) + d(p_{i_0}, h(p_{i_0})) + d(h(p_{i_0}), h(p)) \\ &< \frac{\delta}{4} + \frac{\delta}{2} + d(p_{i_0}, p) < \frac{\delta}{4} + \frac{\delta}{2} + \frac{\delta}{4} = \delta. \end{aligned}$$

Thus for each point $p \in M$, $d(p, h(p)) < \delta \leqq d(p, C(p))$. Then by Lemma 1, $h$ must be identity. So our assertion is proved. q.e.d.

Next, we will calculate the number $k$ for compact manifolds with the sectional curvature satisfying $-b^2 \leqq K_\sigma \leqq 0$ for some $b > 0$.

Let $H$ be the $n$-dimensional hyperbolic space of constant curvature $-b^2$ represented on the Euclidean disk $\{x \in R^n : \|x\|^2 < 1/b^2\}$ with the Riemannian metric $ds^2 = \dfrac{4}{(1 - b^2 \|x\|^2)^2} \sum_{i=1}^{n} dx_i^2$. For a positive number $t$, $[t]$ denotes the integer part of $t$. Let $r, s, r > s$ be any positive numbers.

**Lemma 2.** *For the point* $\bar{p}=(0,\cdots,0)\in H$, *there exists a finite number of points* $\bar{p}_1,\cdots,\bar{p}_l\in B_r(\bar{p})$ *such that*

$$\bigcup_{i=1}^{l} B_s(\bar{p}_i)\supset\overline{B_r(\bar{p})}.$$

*The number* $l$ *can be taken as*

$$l\leqq\left(\left[\frac{r}{s}\right]+1\right)\left(\left[\frac{2R}{\alpha}\right]+1\right)^n+1$$

*where* $R$ *and* $\alpha$ *are detemined with* $\beta$ *and* $a$ *as follows*:

$$(**)\qquad \begin{cases}\cosh b\beta=\cosh bs/\cosh\dfrac{bs}{2}, & a=\tanh\dfrac{br}{2}/b,\\ R=2a/(1-a^2b^2), & \alpha=\left\{\dfrac{2R^2}{n}\left(1-\cos\dfrac{\beta}{R}\right)\right\}^{1/2}.\end{cases}$$

**Proof.** Let $\gamma:[0,\infty)\to H$ be a geodesic starting from $\bar{p}$. Consider a covering of $\gamma([0,r])$:

$$\bigcup_{j=0}^{[r/s]+1} B_s(\gamma(j\cdot s))\supset\gamma([0,r]).$$

Let $\beta$ be the number satisfying $(**)$. Then, by using the hyperbolic trigonometry, we can see that $\beta$-tubular neighborhood

$$U_\beta(\gamma([0,r])):=\{\bar{q}\in H: d(\bar{q},\gamma([0,r]))<\beta\}$$

of $\gamma$ is contained in $\bigcup_j B_s(\gamma(j\cdot s))$. And we have

$$U_\beta(\gamma([0,r]))\cap\partial B_r(\bar{p})\supset B^{n-1}_{\beta'}(\gamma(r))\supset B^{n-1}_{\beta}(\gamma(r))$$

for some $\beta'>\beta$, where $B^{n-1}_{\beta'}(\gamma(r))$ denotes the open metric ball in $\partial B_r(\bar{p})$ with radius $\beta'$ centered at $\gamma(r)$. Note that $\partial B_r(\bar{p})$ is isometric to the standard $(n-1)$-dimensional sphere $S^{n-1}(R)$. For any point $\bar{q}\in B^{n-1}_{\beta}(\gamma(r))$ and the geodesic $\eta:[0,r]\to H$ connecting from $\gamma(0)$ to $\bar{q}$, we easily see that $\eta([0,r])\subset U_\beta(\gamma([0,r]))$. Thus, if we can find a finite number of points $\bar{p}_0,\cdots,\bar{p}_m\in S^{n-1}(R)$ such that

$$\bigcup_{i=1}^{m} B^{n-1}_{\beta'}(\bar{p}_i)\supset S^{n-1}(R),$$

then, from the above fact, we have

$$\overline{B_r(\bar{p})}\subset\bigcup_{i=1}^{m} U_\beta(\gamma_i([0,r]))\subset\bigcup_{i=1}^{m}\bigcup_{j=0}^{[r/s]+1} B_s(\gamma_i(j\cdot s)),$$

where $\gamma_i:[0,r]\to H$ is the geodesic from $\bar{p}$ to $\bar{p}_i$, $i=1,\cdots,m$.

Now, for any point $p=(p_1,\cdots,p_n)\in R^n$ and a positive number $\rho>0$, we set

$$I(p,\rho):=\{x=(x_1,\cdots,x_n)\in R^n: 0<|x_i-p_i|<\rho/2, 1\leqq i\leqq n\}.$$

Then we can easily see

$$\sup\{d(q,q'): q,q'\in I(p,\rho)\cap S^{n-1}(R)\}$$
$$\leqq R\arccos\frac{2R^2-n\rho^2}{2R^2}\qquad\text{if } n\rho^2\leqq 2R^2.$$

where $d$ is the distance function on $S^{n-1}(R)$. We choose $\alpha>0$ such that

$$R\arccos\frac{2R^2-n\alpha^2}{2R^2}=\beta$$

i.e. $\alpha=\left\{\frac{2R^2}{n}\left(1-\cos\frac{\beta}{R}\right)\right\}^{1/2}$.

Thus, if we can find a finite number of points $q_1, \cdots, q_m \in R^n$ such that

$$\bigcup_{i=1}^{m} \overline{I(q_i, \alpha)} \supset S^{n-1}(R) \quad \text{and} \quad \overline{I(q_i, \alpha)} \cap S^{n-1}(R) \neq \emptyset, i=1, 2, \cdots, m$$

and taking a point $\bar{p}_i \in \overline{I(q_i, \alpha)} \cap S^{n-1}(R)$ for each $i$, then, from the above fact, we have

$$S^{n-1}(R)=\bigcup_{i=1}^{m} \overline{I(q_i, \alpha)} \cap S^{n-1}(R) \subset \bigcup_{i=1}^{m} \overline{B_\beta^{n-1}(\bar{p}_i)} \subset \bigcup_{i=1}^{m} B_{\beta'}^{n-1}(\bar{p}_i).$$

Hence, by means of an Euclidean geometric consideration, we can choose an $m$ with the above mentioned property such that

$$m \leqq \begin{cases} \left(\left[\frac{2R}{\alpha}\right]+1\right)^n & \text{if } \left[\frac{2R}{\alpha}\right] \neq \frac{2R}{\alpha} \\ \left[\frac{2R}{\alpha}\right]^n & \text{if } \left[\frac{2R}{\alpha}\right] = \frac{2R}{\alpha}. \end{cases} \qquad \text{q.e.d.}$$

**Remark.** The estimation of the number $m$ in the proof of Lemma 2 is very rough. We counted the number of points $q_i \in R^n$ such that $\overline{I(q_i, \alpha)} \cap S^{n-1}(R)=\emptyset$. When $\alpha$ is small with respect to $R$, then, for example, we can choose an upper bound of $m$ as

$$m \leqq \begin{cases} \left\{2\left(\left[\frac{R}{\alpha}\right]+1\right)\right\}^n - \left\{2\left[\frac{R}{\sqrt{n}\alpha}\right]\right\}^n & \text{if } \left[\frac{R}{\alpha}\right] \neq \frac{R}{\alpha} \\ \left(2\left[\frac{R}{\alpha}\right]\right)^n - \left(2\left[\frac{R}{\sqrt{n}\alpha}\right]\right)^n & \text{if } \left[\frac{R}{\alpha}\right] = \frac{R}{\alpha}. \end{cases}$$

And it will be possible to get a more sharp estimation.

Using this Lemma, we have

**Proposition 2.** *Let $M$ be an $n$-dimensional compact Riemannian manifold with the diameter $d$ and the injectivity radius $\delta$, whose sectional curvature $K_\sigma$ satisfies $-b^2 \leqq K_\sigma \leqq 0$ for some $b>0$. Then there exists a finite number of points $p_1, \cdots, p_k \in M$ such that*

$$\bigcup_{i=1}^{k} B_{\delta/4}(p_i)=M \quad \textit{with} \quad k \leqq \left(\left[\frac{4d}{\delta}\right]+1\right)\left(\left[\frac{2R}{\alpha}\right]+1\right)^n+1$$

*where $R$ and $\alpha$ are determined as in Lemma 2 by replacing $r$ and $s$ with $d$ and $\delta/4$ respectively.*

**Proof.** Let $\pi: \tilde{M} \to M$ be the universal covering of $M$ and $\pi$ its projection. Then as is well known, $\tilde{M}$ is diffeomorphic to an $n$-dimensional Euclidean space and each geodesic segment of $M$ is the shortest connection between its end points. Fix a point $p \in M$ and $\tilde{p} \in \{\pi^{-1}(p)\}$. Clearly $B_d(\tilde{p})$ contains a fundamental domain $D$ of this covering i.e. $\pi: D \to M$ is one-to-one and $\pi: \bar{D} \to M$ is onto. Since $\delta$ is the injectivity radius of $M$, for each point $\tilde{q} \in \tilde{M}$, $\pi: B_{\delta/4}(\tilde{q}) \to B_{\delta/4}(\pi(\tilde{q}))$ is onto and isometric. So, if we can find a finite number of points $\tilde{p}_1, \cdots, \tilde{p}_k \in B_d(\tilde{p})$ satisfying $\bigcup_{i=1}^{k} B_{\delta/4}(\tilde{p}_i) \supset \overline{B_d(\tilde{p})}$, then we have $\bigcup_{i=1}^{k} B_{\delta/4}(\pi(\tilde{p}_i))=M$. Let $H$ and

$\bar{p} \in H$ be the space and the point as in Lemma 2. Let $l: T_p(H) \to T_{\bar{p}}(\tilde{M})$ be an isometry. And consider a mapping

$$\varphi := \exp_{\bar{p}} \cdot l \cdot \exp_{\bar{p}}^{-1} : H \to \tilde{M}.$$

For any point $\bar{q} \in B_d(\bar{p})$, let $\gamma : [0, \delta/4] \to H$ be a geodesic starting from $\bar{q}$. Then, by Rauch's Comparison Theorem (see [2: Korollar p. 179]),

$$L(\gamma) \geqq L(\varphi(\gamma))$$

where $L$ denotes the length of a curve. So $\varphi(B_{\delta/4}(\bar{q})) \subset B_{\delta/4}(\varphi(\bar{q}))$ for any $\bar{q} \in B_d(\bar{p})$. Thus, if we find a finite number of points $\bar{p}_1, \cdots, \bar{p}_k \in H$ such that $\bigcup_{i=1}^k B_{\delta/4}(\bar{p}_i) \supset \overline{B_d(\bar{p})}$, then, from the above facts, we have

$$\overline{B_d(\bar{p})} = \overline{\varphi(B_d(\bar{p}))} \subset \bigcup_{i=1}^k \varphi(B_{\delta/4}(\bar{p}_i)) \subset \bigcup_{i=1}^k B_{\delta/4}(\varphi(\bar{p}_i)).$$

And the existence of such points $\bar{p}_i$ is already shown in Lemma 2.

q.e.d.

Summerising the above,

**Theorem.** *Let $M$ be an $n$-dimensional compact connected Riemannian manifold with non-positive sectional curvature $K_\sigma$; $-b^2 \leqq K_\sigma \leqq 0$ for some $b > 0$ and with Ricci curvature negative at some point. Let $d$ and $\delta$ be the diameter and the injectivity radius of $M$ respectively. Then*

$$\textit{the order of } I(M) \leqq k^k,$$

*where $k$ is given by $k = \left(\left[\frac{4d}{\delta}\right] + 1\right)\left(\left[\frac{2R}{\alpha}\right] + 1\right)^n + 1$ and $R, \alpha$ are determined with $\beta, \alpha$ as follows:*

$$\cosh b\beta = \cosh \frac{b\delta}{4} \Big/ \cosh \frac{b\delta}{8}, \qquad a = \tanh \frac{bd}{2} \Big/ b,$$

$$R = 2a/(1 - a^2 b^2), \qquad \alpha = \left\{\frac{2R^2}{n}\left(1 - \cos \frac{\beta}{R}\right)\right\}^{1/2}.$$

## References


[1] T. Frankel: On Theorems of Hurewitz and Bochner. J. Math. Mech., **15**, 373–377 (1966).

[2] D. Gromoll, W. Klingenberg, and W. Meyer: Riemannsche Geometrie im Grossen. Springer-Verlag (1968).

[3] H. C. Im Hof: Über die Isometriegruppe einer kompacten Mannigfaltigkeiten negative Krummung. Comm. Math. Helv., **48**, 14–30 (1973).

[4] H. Huber: Über die Isometriegruppe einer kompakten Mannigfaltigkeit mit negativer Krummung. Helv. Phys. Acta, **45**, 277–288 (1972).

[5] H. B. Lawson and S. T. Yau: Compact manifolds of nonpositive curvature. J. Diff. Geom., **7**, 211–228 (1972).