167. A Remark on the Sobolev Inequality for Riemannian Submanifolds

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Recently, D. Hoffman and J. Spruck proved a Sobolev inequality in [2] as follows:

Let $M \to \bar{M}$ be an isometric immersion of Riemannian manifolds of dimension m and n, respectively. Using the following quantities:

$\bar{K}_\pi$ = sectional curvature for plane section π in $\bar{M}$,

H = mean curvature vector field of the immersion,

$\bar{R}(M)$ = minimum distance for the cut locus in $\bar{M}$ for all points in M,

ω_m = volume of the unit ball in R^m

and

b = a positive real number

and assuming $\bar{K}_\pi < b^2$, then for any non-negative C^1 function h on M with compact support and $h|\partial M \equiv 0$ we have

$$(1) \qquad \left(\int_M h^{m/(m-1)} dV_M\right)^{(m-1)/m} \leqq c(m) \int_M [|\nabla h| + h|H|] dV_M,$$

provided

$$(2) \qquad b\left\{\frac{1}{(1-\alpha)\omega_m} \operatorname{Vol}(\operatorname{supp} h)\right\}^{1/m} \leqq 1$$

and

$$(3) \qquad \rho_* := \frac{1}{b} \sin^{-1}\left[b\left\{\frac{1}{(1-\alpha)\omega_m} \operatorname{Vol}(\operatorname{supp} h)\right\}^{1/m}\right] \leqq \frac{1}{2}\bar{R}(M),$$

where α is a free parameter, $0<\alpha<1$, and

$$(4) \qquad c(m) = c(m, \alpha) := \frac{\pi}{2} \cdot \frac{2^{m-2}}{\alpha} \cdot \frac{m}{m-1} \cdot \left\{\frac{1}{(1-\alpha)\omega_m}\right\}^{1/m}.$$

This inequality is very important from the geometric point of view, since this type of inequalities will have a number of interesting applications in differential geometry. In this short paper, we will show that $c(m)$ in (1) must be revised by a more sharper constant, for example

$$(4') \qquad c'(m) = c'(m, \alpha, t) := \frac{\pi}{2} \cdot \frac{(m-\alpha)t^{m-1} - (1-\alpha)}{(m-1)\alpha} \cdot \frac{m}{m-1} \cdot \left\{\frac{1}{(1-\alpha)\omega_m}\right\}^{1/m},$$

provided (2) and

$$(3') \qquad t\rho_* \leqq \bar{R}(M),$$

where $0<\alpha<1$ and $2 \leqq t$.

As is shown in [2], the inequality (1) is implied from Lemma 4.2. But, in the proof of this lemma, they made some elementary mistakes in integral calculation, and so their Sobolev and isoperimetric inequalities must be revised in constants $c(m)$ and so forth. In place of Lemma 4.2 in [2] for the real b case, we shall give the following

Lemma. *Let $\xi \in M$ be such that $h(\xi) \geqq 1$. Let α, t satisfy $0<\alpha<1<t$. Set*

$$\rho_0 := \frac{1}{b} \sin^{-1}\left[b\left\{ \frac{1}{(1-\alpha)\omega_m} \int_M h dV_M \right\}^{1/m} \right], \tag{5}$$

provided

$$b\left\{ \frac{1}{(1-\alpha)\omega_m} \int_M h dV_M \right\}^{1/m} \leqq 1. \tag{6}$$

Then, there exists a ρ, $0<\rho<\rho_0$, such that

$$\bar{\phi}_\xi(t\rho) \leqq \frac{(m-\alpha)t^{m-1}-(1-\alpha)}{(m-1)\alpha} \rho_0 \bar{\Psi}_\xi(\rho) \tag{7}$$

provided

$$t\rho_0 \leqq \bar{R}(M). \tag{8}$$

In the statement of the lemma, $\bar{\phi}_\xi$ and $\bar{\Psi}_\xi$ are defined by

$$\bar{\phi}_\xi(\rho) := \int_{M \cap B_\rho(\xi)} h dV_M, \tag{9}$$

$$\bar{\Psi}_\xi(\rho) := \int_{M \cap B_\rho(\xi)} [|\nabla h| + h|H|] dV_M, \tag{10}$$

and $B_\rho(\xi)$ is the geodesic ball in $\bar{M}$ with center ξ and radius ρ. (7) can be replaced with a more simpler but duller one:

$$\bar{\phi}_\xi(t\rho) \leqq \frac{t+\alpha-t\alpha}{\alpha} t^{m-1} \rho_0 \bar{\Psi}_\xi(\rho). \tag{7'}$$

Proof of Lemma. As is stated in the proof of Lemma 4.2 in [2], we have

$$(\sin b\sigma)^{-m}\bar{\phi}_\xi(\sigma-\varepsilon) \leqq (\sin b\rho_0)^{-m}\bar{\phi}_\xi(\rho_0) + \int_0^{\rho_0} (\sin b\rho)^{-m}\bar{\Psi}_\xi(\rho)d\rho \tag{11}$$
for all ε and σ, $0<\varepsilon<\sigma<\rho_0$.

In place of (7), we set

$$\bar{\phi}_\xi(t\rho) \leqq \frac{\lambda t^{m-1}}{\alpha} \rho_0 \bar{\Psi}_\xi(\rho), \tag{12}$$

where λ is a constant depending on α, t and m determined afterwards.

Suppose there exists no ρ, $0<\rho<\rho_0$, satisfying (12), namely that

$$\bar{\Psi}_\xi(\rho) < \frac{\alpha}{\lambda t^{m-1}\rho_0} \bar{\phi}_\xi(t\rho) \qquad \text{for all } \rho \in (0, \rho_0). \tag{13}$$

Then, by changing of the integral parameters, we have easily

$$\begin{aligned} \int_0^{\rho_0} (\sin b\rho)^{-m}\bar{\Psi}_\xi(\rho)d\rho &< \frac{\alpha}{\lambda t^{m-1}\rho_0} \int_0^{\rho_0} (\sin b\rho)^{-m}\bar{\phi}_\xi(t\rho)d\rho \\ &= \frac{\alpha}{\lambda t^m \rho_0}\left[\int_0^{\rho_0} \left(\sin \frac{b\rho}{t}\right)^{-m} \bar{\phi}_\xi(\rho)d\rho + \int_{\rho_0}^{t\rho_0} \left(\sin \frac{b\rho}{t}\right)^{-m} \bar{\phi}_\xi(\rho)d\rho \right]. \end{aligned} \tag{14}$$

Since we have

$$\sin\frac{\theta}{t}>\frac{1}{t}\sin\theta \qquad \text{for } 0<\theta<\pi,\ t>1$$

and $0<b\rho<\pi/2$ for $\rho\in(0,\rho_0)$, we obtain

$$\int_0^{\rho_0}\left(\sin\frac{b\rho}{t}\right)^{-m}\bar{\phi}_\xi(\rho)d\rho<t^m\int_0^{\rho_0}(\sin b\rho)^{-m}\bar{\phi}_\xi(\rho)d\rho \le t^m\rho_0\sup_{\sigma\in(0,\rho_0)}(\sin b\sigma)^{-m}\bar{\phi}_\xi(\sigma). \tag{15}$$

Next, making use of the fact that the function $\sin(b/t)\rho$ of ρ is convex upward for $\rho\in(0,t\rho_0)$, we have

$$\sin\frac{b\rho}{t}>\frac{\rho}{t\rho_0}\sin b\rho_0 \qquad \text{for } 0<\rho<t\rho_0,$$

hence we obtain

$$\int_{\rho_0}^{t\rho_0}\left(\sin\frac{b\rho}{t}\right)^{-m}d\rho<\left(\frac{t\rho_0}{\sin b\rho_0}\right)^m\int_{\rho_0}^{t\rho_0}\frac{d\rho}{\rho^m}=\frac{t(t^{m-1}-1)}{m-1}\cdot\frac{\rho_0}{(\sin b\rho_0)^m}.$$

Since we have from (9)

$$\bar{\phi}_\xi(\rho)\le\int_M hdV_M, \tag{16}$$

the above inequality implies immediately

$$\int_{\rho_0}^{t\rho_0}\left(\sin\frac{b\rho}{t}\right)^{-m}\bar{\phi}_\xi(\rho)d\rho<\frac{t(t^{m-1}-1)}{m-1}\cdot\frac{\rho_0}{(\sin b\rho_0)^m}\cdot\int_M hdV_M. \tag{17}$$

We may assume $m\ge 2$, then we have

$$\frac{t(t^{m-1}-1)}{m-1}<t^{m-1}(t-1) \qquad \text{for } t>1$$

and hence

$$\int_{\rho_0}^{t\rho_0}\left(\sin\frac{b\rho}{t}\right)^{-m}\bar{\phi}_\xi(\rho)d\rho<t^{m-1}(t-1)\cdot\frac{\rho_0}{(\sin b\rho_0)^m}\cdot\int_M hdV_M. \tag{17'}$$

Now, combining (15) and (17) or (17′) with (14), we get

$$\begin{aligned}\int_0^{\rho_0}(\sin b\rho)^{-m}\bar{\Psi}_\xi(\rho)d\rho&<\frac{\alpha}{\lambda}\sup_{\sigma\in(0,\rho_0)}(\sin b\sigma)^{-m}\bar{\phi}_\xi(\sigma)\\&+\frac{\alpha(t^{m-1}-1)}{(m-1)\lambda t^{m-1}}\cdot\frac{1}{(\sin b\rho_0)^m}\cdot\int_M hdV_M\end{aligned} \tag{18}$$

or

$$\begin{aligned}\int_0^{\rho_0}(\sin b\rho)^{-m}\bar{\Psi}_\xi(\rho)d\rho&<\frac{\alpha}{\lambda}\sup_{\sigma\in(0,\rho_0)}(\sin b\sigma)^{-m}\bar{\phi}_\xi(\sigma)\\&+\frac{\alpha(t-1)}{\lambda t}\cdot\frac{1}{(\sin b\rho_0)^m}\cdot\int_M hdV_M.\end{aligned} \tag{18'}$$

On the other hand, by the definition of ρ_0 we have

$$\frac{1}{(\sin b\rho_0)^m}\cdot\int_M hdV_M=\frac{1}{b^m}(1-\alpha)\omega_m. \tag{19}$$

From (11), we have

$$\sup_{\sigma\in(0,\rho_0)}(\sin b\sigma)^{-m}\bar{\phi}_\xi(\sigma)\le(\sin b\rho_0)^{-m}\bar{\phi}_\xi(\rho_0)+\int_0^{\rho_0}(\sin b\rho)^{-m}\bar{\Psi}_\xi(\rho)d\rho.$$

From this inequality, (16), (19) and (18), we get

$$\left(1-\frac{\alpha}{\lambda}\right)\sup_{\sigma\in(0,\rho_0)}(\sin b\sigma)^{-m}\bar{\phi}_\xi(\sigma)<\left\{1+\frac{\alpha(t^{m-1}-1)}{(m-1)\lambda t^{m-1}}\right\}\cdot\frac{1-\alpha}{b^m}\omega_m.$$

Since h is of C^1 class, we have

$$\sup_{\sigma\in(0,\rho_0)}\frac{b^m\bar{\phi}_\xi(\sigma)}{(\sin b\sigma)^m}\geqq\omega_m.$$

Using this, the above inequality implies

$$1-\frac{\alpha}{\lambda}<(1-\alpha)\left\{1+\frac{\alpha(t^{m-1}-1)}{(m-1)\lambda t^{m-1}}\right\}, \tag{20}$$

provided $\lambda\geqq\alpha$. Therefore, if the positive constant λ satisfies

$$1-\frac{\alpha}{\lambda}\geqq(1-\alpha)\left\{1+\frac{\alpha(t^{m-1}-1)}{(m-1)\lambda t^{m-1}}\right\},$$

i.e.

$$\lambda\geqq\frac{1}{m-1}\left(m-\alpha-\frac{1-\alpha}{t^{m-1}}\right), \tag{21}$$

then we reach a contradiction. Hence, setting

$$\lambda:=\frac{1}{m-1}\left(m-\alpha-\frac{1-\alpha}{t^{m-1}}\right), \tag{22}$$

we have

$$\frac{\lambda t^{m-1}}{\alpha}=\frac{(m-\alpha)t^{m-1}-(1-\alpha)}{(m-1)\alpha}.$$

Thus, (7) must be true for some ρ, $0<\rho<\rho_0$. Q.E.D.

Here, we shall give an analogous formula to (7), which is derived from the argument using (18′) in place of (18). From (11), (16), (19) and (18′) we get

$$\left(1-\frac{\alpha}{\lambda}\right)\sup_{\sigma\in(0,\rho_0)}(\sin b\sigma)^{-m}\bar{\phi}_\xi(\sigma)<\left\{1+\frac{\alpha(t-1)}{\lambda t}\right\}\cdot\frac{1-\alpha}{b^m}\omega_m$$

and hence

$$1-\frac{\alpha}{\lambda}<(1-\alpha)\left\{1+\frac{\alpha(t-1)}{\lambda t}\right\}, \tag{20′}$$

provided $\lambda\geqq\alpha$. Therefore, if the positive constant λ satisfies

$$1-\frac{\alpha}{\lambda}\geqq(1-\alpha)\left\{1+\frac{\alpha(t-1)}{\lambda t}\right\},$$

i.e.

$$\lambda\geqq\frac{(2-\alpha)t-(1-\alpha)}{t}, \tag{21′}$$

then we reach a contradiction. Hence, setting

$$\lambda:=\frac{(2-\alpha)t-(1-\alpha)}{t}, \tag{22′}$$

we have

$$\frac{\lambda t^{m-1}}{\alpha}=\frac{(2-\alpha)t-(1-\alpha)}{\alpha}t^{m-2}.$$

Thus, we have

$$\bar{\phi}_\varepsilon(t\rho) \leqq \frac{(2-\alpha)t-(1-\alpha)}{\alpha} t^{m-2} \rho_0 \bar{\Psi}_\varepsilon(\rho) \tag{7''}$$

and (7′) for some ρ, $0<\rho<\rho_0$, since

$$(2-\alpha)t-(1-\alpha)<t(t+\alpha-t\alpha).$$

Last, we state an isoperimetric inequality for Riemannian submanifolds, which is derived from (1) replaced $c(m)$ by $c'(m)$, in a revised form of the one in [2].

Theorem. *Let M be a compact submanifold with $\partial M \neq \phi$ in a Riemannian manifold $\bar{M}$ and assume $\bar{K}_\pi \leqq b^2$, $b>0$. Then, for $0<\alpha<1$, $t \geqq 2$, we have*

$$(\mathrm{Vol}\,(M))^{(m-1)/m} < c'(m,\alpha,t) \left(\mathrm{Vol}\,(\partial M) + \int_M |H|\, dV_M\right), \tag{23}$$

provided

$$b\left\{\frac{1}{(1-\alpha)\omega_m} \mathrm{Vol}\,(M)\right\}^{1/m} \leqq 1 \tag{24}$$

and

$$\frac{t}{b} \sin^{-1}\left[b\left\{\frac{1}{(1-\alpha)\omega_m} \mathrm{Vol}\,(M)\right\}^{1/m}\right] \leqq \bar{R}(M), \tag{25}$$

where $m = \dim M$.

References

[1] J. H. Michael and I. M. Simon: Sobolev and mean-value inequalities on generalized submanifolds of R^n. Comm. Pure and Appl. Math., **26**, 361–379 (1973).

[2] D. Hoffman and J. Spruck: Sobolev and isoperimetric inequalities for Riemannian submanifolds. Comm. Pure and Appl. Math., **27**, 715–727 (1974).