3. *Riemannian Manifolds Admitting Some Geodesic. II*[*)]

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1. Introduction. In this note we shall show some types of manifold which admits an isometry invariant geodesic. Let M be a Riemannian manifold and f be an isometry of M, then a geodesic α is called f-invariant geodesic if $f\alpha=\alpha$. The problem of the existence of isometry invariant geodesic is proposed by K. Grove ([1]) and in case of connected, simply connected and compact manifold a criterion of this problem is shown by him. We reform it into a calculable form ([3]) and here actually show some types of manifold which admits an invariant geodesic. The existence of an invariant geodesic is known for all compact connected, simply connected orientable manifolds of odd dimension and for its orientation preserving isometry ([3]). And so here we are concerned with even dimensional manifold.

Here the manifold M which we consider is following

[H] *compact, connected and orientable and its fundamental group is finite.*

An order of an isometry f is defined by the minimal integer n such that f^n is homotopic to the identity and denoted by $\mathrm{ord}(f)$. And a rank of the k-th homology group $H_k(M, Z)$ over the integer group Z is denoted by rank $H_k(M, Z)$. Then our main results are following;

Theorem A. *Let M be a $2k$-dimensional manifold of* [H] *and f be an orientation preserving isometry* ($k>1$). *If rank* $H_k(M,Z)=2$ *and* $\mathrm{ord}(f)\not\equiv 0 \pmod 3$, *then there exists an f-invariant geodesic. If rank* $H_k(M,Z)=3$, $\mathrm{ord}(f)\not\equiv 0 \pmod 2$ *and f has no eigenvalue* 1, *then there exists an f-invariant geodesic.*

Theorem B. *Let M be a $2k$-dimensional manifold of [H] and f be an orientation preserving isometry* ($k>1$). *If rank* $H_k(M,Z)=$ *even and* $\mathrm{ord}(f)=2$, 4 *or* 8, *then there exists an f-invariant geodesic.*

For two dimensional manifold of [H] we can prove that there exists an f-invariant geodesic for each orientation preserving isometry f by using our result of [3].

2. Lemmas. Let M be a Riemannian manifold of $2k$-dimension and f be an isometry of M. Then a trace of an induced homomorphism $f_k\colon H_k(M,Z)\to H_k(M,Z)$ of k-th homology group which is defined by a

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trace $\bar{f}_k: H_k(M,Z)/\text{torsion}\to H_k(M,Z)/\text{torsion}$ is denoted by $\lambda_k(f)$ and so the trace $\lambda_k(f)$ is an integer. Sometimes we say an eigenvalue of f instead of an eigenvalue of $\bar{f}_k$.

Lemma 1. *Let M be a $2k$-dimensional manifold of* [H] *and f be an orientation preserving isometry. If rank $H_k(M,Z)=2r$ where r is an integer, then there exist r-angles $(\theta_1,\theta_2,\theta_3,\cdots,\theta_r)$ such that $\lambda_k(f)=2(\cos\theta_1+\cos\theta_2+\cos\theta_3+\cdots+\cos\theta_r)$ where $\theta_i=2t\pi/\text{ord}(f)$ for some t $(1\leqslant i\leqslant r,\ t=0,\pm1,\pm2,\cdots)$. If rank $H_k(M,Z)=2r+1$ where r is an integer, then there exist r-angles $(\theta_1,\theta_2,\theta_3,\cdots,\theta_r)$ such that $\lambda_k(f)=2(\cos\theta_1+\cos\theta_2+\cos\theta_3+\cdots+\cos\theta_r)\pm1$ where $\theta_i=2t\pi/\text{ord}(f)$ for some t $(1\leqslant i\leqslant r,t=0,\pm1,\pm2,\cdots)$. Moreover the angles are determined uniquely for the homotopy class of f.*

Proof. We have only to show this lemma in case of $\Pi_1(M)=1$. For in case of $\Pi_1(M)\neq1$ we consider a compact covering space $\tilde{M}$ of M with $\Pi_1(\tilde{M})=1$ because $\Pi_1(M)$ is finite. Let $G(M)$ be a group of isometries of M and $G_0(M)$ be a component of the identity, then it is known that the order of $G(M)/G_0(M)$ is finite. And so there exists a positive integer n such that f^n is homotopic to the identity. Since $\bar{f}_k^n=1:\bar{H}_k(M,Z)\to\bar{H}_k(M,Z)$ where $\bar{H}_k(M,Z)=H_k(M,Z)/\text{torsion}$, $\sigma^n=1$ for an eigenvalue σ of the integral matrix $\bar{f}_k$. Let σ_j $(1\leqslant j\leqslant m)$ be the eigenvalues of $\bar{f}_k$ where $m=\text{rank}\,H_k(M,Z)$, then we have $\sigma_1\sigma_2\sigma_3\cdots\sigma_m=\pm1$ because the determinant of $\bar{f}_k$ is ±1. Thus we have $\sigma_j=\cos\theta_j+\sqrt{-1}\sin\theta_j$ where $n\theta_j=2t\pi$ $(t=0,\pm1,\pm2,\cdots)$ and hence

$$\lambda_k(f)=\sigma_1+\sigma_2+\sigma_3+\cdots+\sigma_m=2(\cos\theta_1+\cos\theta_2+\cos\theta_3+\cdots+\cos\theta_r)$$

if $m=2r$ and

$$\lambda_k(f)=\sigma_1+\sigma_2+\sigma_3+\cdots+\sigma_m=2(\cos\theta_1+\cos\theta_2+\cos\theta_3+\cdots+\cos\theta_r)\pm1$$

if $m=2r+1$. And the last half will be proved by using next lemma.

If two isometries f and g are homotopic and suppose that $\text{ord}(f)=p>q=\text{ord}(g)$, then we have f^q homotopic to the identity. This is impossible and so we conclude $\text{ord}(f)=\text{ord}(g)$. Thus we have

Lemma 2. *Let f and g be two isometries of a manifold of* [H]. *If f is homotopic to g, then* $\text{ord}(f)=\text{ord}(g)$.

Hence for the homotopy class of f the angles $(\theta_1,\theta_2,\theta_3,\cdots,\theta_r)$ of Lemma 1 are determined uniquely.

3. Proof of main results. In the previous paper ([3]) we have the following theorem;

Theorem C. *Let M be a $2k$-dimensional manifold $(k>1)$ of* [H] *and f be an orientation preserving isometry of M. Then there exists an f-invariant geodesic if $\lambda_k(f)=$ even.*

If rank $H_k(M,Z)=2$, then $\lambda_k(f)=2\cos\theta$ by Lemma 1. Since $\lambda_k(f)$ is an integer, $\lambda_k(f)=0,\pm1$ or ±2 and so we have only to remove the case of $\lambda_k(f)=\pm1$. If $\text{ord}(f)\not\equiv0\ (\text{mod}\ 3)$, $\lambda_k(f)\neq\pm1$ because of

$\theta=2t\pi/\text{ord}(f)$ $(t=0, \pm 1, \pm 2, \cdots)$. If rank $H_k(M, Z)=3$, then $\lambda_k(f)=2\cos\theta\pm 1$ by Lemma 1. If ord $(f)\not\equiv 0 \pmod 2$ and if f has no eigenvalue of 1, then $\lambda_k(f)=0$ or ± 2 because $\lambda_k(f)$ is an integer and $\theta=2t\pi/\text{ord}(f)$ $(t=0, \pm 1, \pm 2, \cdots)$. And hence in both cases there exists an f-invariant geodesic by Theorem C. These are the complete proof of Theorem A.

Now we shall show theorem B. If rank $H_k(M, Z)=2r$, then $\lambda_k(f)=2(\cos\theta_1+\cos\theta_2+\cos\theta_3+\cdots+\cos\theta_r)$ and $\sigma_i=\cos\theta_i+\sqrt{-1}\sin\theta_i$ $(1\leqslant i\leqslant r)$ is a solution of $\sigma^n=1$ where $n=2, 4$ or 8. Then $\cos\theta_i=0, \pm 1$ or $\pm 1/\sqrt{2}$ $(1\leqslant i\leqslant r)$ and hence $\lambda_k(f)=\text{even}$. Thus there exists an f-invariant geodesic by Theorem C. These complete the proof of Theorem B.

References

[1] Grove, K.: Condition (C) for the Energy Integral on Certain Path-Spaces and Applications to the Theory of Geodesics (preprint series of Aarhus University (1970)).

[2] Kobayashi, S.: Transformation Groups in Differential Geometry. Ergebnisse der Math., Bd, 70 (1972).

[3] Kurogi, T.: Riemannian manifolds admitting some geodesic. Proc. Japan Acad., **50** (1974).