1. The Exact Functor Theorem for BP_*/I_n-Theory

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§ 1. Let p be a prime number and $BP_*(-)$ be the Brown-Peterson homology theory with the coefficient $BP_* \simeq Z_{(p)}[v_1, \cdots]$. Landweber proved the following theorem [4].

Exact functor theorem. *Let G be a BP_*-module and $I_n=(p, v_1, \cdots, v_n)$ be the ideal of BP_* generated by $p, v_1, \cdots, v_n$. Then if the homomorphism*

$$v_{n+1}: G/I_nG \rightarrow G/I_nG, \qquad v_{n+1}(g)=v_{n+1}\cdot g,$$

is monic for each $n \geqq -1$, then $BP_*(-)\underset{BP_*}{\otimes} G$ is a homology theory.

On the other hand Sullivan-Baas constructed bordism theories with singularities (Math. Scan. **33**, 1973). Analogously we can define the homology theory $BP(I_n)_*(-)$ with the coefficient $BP_*/I_n \simeq Z_p[v_{n+1}, \cdots]$ [2], [8]. In this paper we shall prove the exact functor theorem for $BP(I_n)_*$-theory.

Theorem. *Let G be a BP_*/I_n-module. If the homomorphism*

$$v_{m+1}: G/I_mG \rightarrow G/I_mG$$

is monic for each $m \geqq n$, then $BP(I_n)_*(-)\underset{BP_*/I_n}{\otimes} G$ is a homology theory.

Remark. We always consider *reduced* homology theories in the category of *finite CW*-complexes.

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§ 2. First, we take argument in the *cohomology theory $BP(I_n)^*(-)$* which is the *Spanier-Whitehead dual* to $BP(I_n)_*(-)$.

Lemma 1. *Let M be a finitely generated BP^*/I_n- and $BP(I_n)^*(BP(I_n))$-module. Then there exists a BP^*/I_n-filtration such that*

(1) $$M=M_0 \supset M_1 \supset \cdots \supset M_k=\{0\}$$

(2) $$M_s/M_{s+1} \simeq BP^*/J_s \qquad \text{for } 0 \leqq s < k$$

where J_s is an (invariant) ideal of BP^* satisfying $\theta(J_s) \subset J_s$ for any operation $\theta \in BP^*(BP)$.

Proof. For each $\theta \in BP^*(BP)$, let $\bar{\theta}_n$ be the set of $\theta_n \in BP(I_n)^*(BP(I_n))$ which commute the following diagram.

$$\begin{array}{ccc} BP & \xrightarrow{i} & BP(I_n) \\ \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \theta_n} \\ S^mBP & \xrightarrow{i} & S^mBP(I_n) \end{array}$$

where i is the natural inclusion map. Since $BP(I_n)^*(BP(I_n)) \simeq BP^*/I_n \hat{\otimes} R \otimes \Lambda[Q_0, \cdots, Q_n]$ and R is a $Z_{(p)}$-free module which is generated by $\{(r_\alpha)_n\}$ ([2], [7], [8]), if $\theta_1, \theta_2 \in \bar{\theta}_n$, we can write $\theta_1 - \theta_2 = \sum_\gamma l_\gamma \cdot \Phi_\gamma$ where $l_\gamma \in BP^*/I_n$, $\Phi_\gamma \in R \otimes \Lambda[Q_0, \cdots, Q_n]$. Since Φ_γ contains at least one Q_i, we have $\dim \Phi_\gamma > 0$. Since $r_\alpha k = \sum_{\alpha=\beta+\gamma} r_\beta(k) \cdot r_\gamma$ (Cartan formula) for $r_\alpha \in BP^*(BP)$ and $k \in BP^*$, $\overline{(r_\alpha k)}_n$ contains $(r_\alpha)_n k$ and $\sum_{\alpha=\beta+\gamma} r_\beta(k) \cdot (r_\gamma)_n$. Then we have

$$(r_\alpha)_n k = \sum_{\alpha=\beta+\gamma} r_\beta(k) \cdot (r_\gamma)_n + \sum l_\gamma(k) \Phi_\gamma(k).$$

Now let $[\sigma_0, \cdots, \sigma_m]$ be a system of BP^*/I_n-module generators of M ($\dim \sigma_i \geqq \dim \sigma_{i+1}$ for $0 \leqq i < m$). If $k\sigma_0 = 0$, we get

$$0 = (r_\alpha)_n(k\sigma_0) = \sum_{\alpha=\beta+\gamma} r_\beta(k)(r_\gamma)_n(\sigma_0) + \sum l_\gamma(k)\Phi_\gamma(k)(\sigma_0)$$
$$= r_\alpha(k)\sigma_0 \qquad \text{(by dimensional reason).}$$

This shows the anihilator ideal $A(\sigma_0)$ of σ_0 is an invariant ideal. Take $M_1 = [0]$ and $M_0 = BP^*/I_n \cdot \sigma_0 \subset M$, then $M_0/M_1 \simeq BP^*/A(\sigma_0)$ is a $BP(I_n)^*(BP(I_n))$-module, so is M/M_0. By induction we can complete the proof.

In the following, suppose G is a BP^*/I_n-module so that the homomorphism

$$v_{m+1}\colon G/I_mG \to G/I_mG, \qquad v_{m+1}(g) = v_{m+1} \cdot g,$$

is monic for each $m \geqq n$.

Lemma 2. *$Tor_1^{BP^*/I_n}(BP^*/I_m, G) = 0$ for all $m \geqq n$.*

Proof. From the short exact sequence,

$$0 \longrightarrow BP^*/I_m \xrightarrow{v_{m+1}} BP^*/I_m \xrightarrow{i} BP^*/I_{m+1} \longrightarrow 0$$

we get the following exact sequence

$$Tor_1^{BP^*/I_n}(BP^*/I_m, G) \xrightarrow{i_1} Tor_1^{BP^*/I_n}(BP^*/I_{m+1}, G) \longrightarrow G/I_mG \xrightarrow{v_{m+1}} G/I_mG$$

Since $v_{m+1}|G/I_mG$ is monic, i_1 is epic. But $Tor_1^{BP^*/I_n}(BP^*/I_n, G) = 0$ clearly, hence we get $Tor_1^{BP^*/I_n}(BP^*/I_m, G) = 0$ for all $m \geqq n$.

Lemma 3. *Let M be a finitely presented BP^*/I_n- and $BP(I_n)^*(BP(I_n))$-module. Then we have $Tor_1^{BP^*/I_n}(M, G) = 0$.*

Proof. If J is a finitely generated invariant ideal, there is a BP_*-filtration such that

$$BP^*/J = B_0 \supset B_1 \supset \cdots \supset B_r = \{0\}, B_i/B_{i+1} \simeq BP^*/I_{n_i}$$

for $n_i \geqq n$ (see [3]). Hence from Lemma 2, $Tor_1^{BP^*/I_n}(BP^*/J, G) = 0$. Now, from Lemma 1, the proof is clear.

Now we proceed to the proof of the exact functor theorem for $BP(I_n)_*$-theory. It is easily seen that $BP(I_n)^*(-) \underset{BP^*/I_n}{\otimes} G$ satisfies the Eilenberg-Steenrod cohomology axioms except for exact and dimension axioms. Let $f^*: BP(I_n)^*(Y) \to BP(I_n)^*(X)$ be the induced map from a continuous map $f: X \to Y$. Since $BP(I_n)^*(X)$ is a finitely presented BP^*/I_n- and $BP(I_n)^*(BP(I_n))$-module, so are ker. f^* and coker. f^*.

From Lemma 3 the exact axiom also satisfies. Then by taking duality, the proof is completed.

§ 3. As an application we get the following Johnson-Wilson theorem for BP_*/I_n, by using the same techniques as [1] [4].

Theorem. *Follows are equivalent for a finite complex X.*

(1) $$\text{hom dim}_{BP_*/I_n} BP(I_n)_*(X) < m-n,$$

(2) $$BP(\cdots, v_n, v_{m+1}, \cdots)_*(X) \simeq BP_*/(v_{m+1}, \cdots) \underset{BP_*}{\otimes} BP(I_n)_*(X),$$

where $BP(\cdots, v_n, v_{m+1}, \cdots)_*(X)$ *is the bordism theory with the coefficient* $BP_*/(\cdots, v_n, v_{m+1}, \cdots) \simeq Z_p[v_{n+1}, \cdots, v_m]$.

Next let $K(n)_*(-)$ be an extraordinary K-theory with the coefficient $Z_p[v_n, v_n^{-1}]$. Then from the theorem we can easily deduced the following Conner-Floyd theorem for BP_*/I_n ([2], [5], [6]).

Theorem (Morava). $BP(I_{n-1})_*(X) \underset{BP_*}{\otimes} Z_p[v_n, v_n^{-1}] \simeq K(n)_*(X)$.

And we can prove the following Hattori-Stong theorem for BP_*/I_n.

Theorem. *Let h be the Hurewicz homomorphism*

$$h: \pi_*(BP(I_{n-1})) \to K(n)_*(BP(I_{n-1})).$$

Then h is a monomorphism.

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