15. A Note on n-movability and S^k-movability*)

By Jin ONO
Department of Mathematics, Tokyo University of Education

(Comm. by Kôsaku YOSIDA, M. J. A., Feb. 12, 1976)

The following problem was raised by K. Borsuk in [4]. "Let S^k denote the k-dimensional sphere. Does there exist a compactum X which is S^k-movable for $k=1,2,\cdots n$, but is not n-movable?" In this paper, we will construct such a continuum X for the case of $n\geqq 2$.

The concepts of n-movability and A-movability were originally given by K. Borsuk in [3] and [4], and they are equivalent to the following definitions. Let X be a compactum and $X=\{X_n, p_{nn'}, N\}$ be an ANR-sequence associated with X, where each X_n, $n\in N$, is a regular ANR-space (see [6]).

Definition 1 (Y. Kodama and T. Watanabe [6]). A compactum X is said to be *k-movable* if for each $n\in N$ there is $n'\in N$, $n'\geqq n$, such that for each $n''\in N$, $n''\geqq n$, and for each compact set $K\subset X_n$, with $\dim K\leqq k$, there is a map $f_{n''}: K\to X_{n''}$ satisfying the homotopy relation: $p_{nn''}f_{n''}\simeq p_{nn'}|_K : K\to X_n$.

Definition 2. Let A be a compactum. A compactum X is said to be *A-movable* if for each $n\in N$ there is $n'\in N$, $n'\geqq n$, such that for each $n''\in N$, $n''\geqq n$, and for each map $f: A\to X_{n'}$, there is a map $f_{n''}: A\to X_{n''}$ satisfying the homotopy relation: $p_{nn''}f_{n''}\simeq p_{nn'}f: A\to X_n$.

The equivalence of the concept of A-movability in Definition 2 and the original one can be shown by the same way as in the proof of Theorem 3 of [5].

Our example is homeomorphic to the continuum constructed by K. Borsuk [2]. For completeness we give its construction. Consider the following compact subsets of an Euclidean 3-space R^3:

$$A_1=\{(x,y,z)\,|\,x^2+y^2+z^2=5, |x|\leqq 2, |z|\leqq 2,\}$$
$$\cup\{(x,y,z)\,|\,x^2+y^2=1, |z|\leqq 2\}$$
$$B_0=\{(x,y,z)\,|\,x^2+y^2+z^2=5, |x|\leqq -2\}$$
$$B_1=\{(x,y,z)\,|\,x^2+y^2+z^2=5, x\geqq 2\}$$
$$A_n=\{(x,y,z)\,|\,(x-4n+4,y,z)\in A_1\}, \qquad n=2,3,\cdots$$
$$B_n=\{(x,y,z)\,|\,(x-4n+4,y,z)\in B_1\}, \qquad n=2,3,\cdots.$$

Put $X_n=B_0\cup A_1\cup A_2\cup\cdots\cup A_n\cup B_n$, for each $n\in N$. For $n,n'\in N$, $n'>n$, define a map $p_{nn'}: X_{n'}\to X_n$ by

*) Dedicated to Professor Kiiti Morita for his 60th birthday.

$$p_{nn'}(x,y,z) = \begin{cases} (x,y,z) & \text{if } x \leqq 4n-2 \\ \left(x, \sqrt{\dfrac{5-(4n-x)^2}{5-(x-4n+4)^2}}\,y, \sqrt{\dfrac{5-(4n-x)^2}{5-(x-4n+4)^2}}\,z\right) & \text{if } 4n-2 \leqq x \leqq 4n-4+\sqrt{5} \\ (4n-4+\sqrt{5}, 0, 0) & \text{if } 4n-4+\sqrt{5} \leqq 4. \end{cases}$$

Then we have an ANR-sequence $X=\{X_n, p_{nn'}, N\}$ consisting of regular ANR-spaces. Note that X_n is a closed orientable 2-manifold of genus n, and $p_{nn'}$ is a map of degree 1, for each $n, n' \in N$, $n'>n$.

Proposition 1. *X is S^k-movable for $k=1,2,3,\cdots$.*

Proof. It is sufficient to show that for each $n \in N$ and each map $f: S^k \to X_n$, there exists a map $g: S^k \to X_n$ such that $f \simeq g$ and $g(S^k) \subset X_n \backslash B_n$. Because for any $n'' \in N$, $n''>n$, $p_{nn''}|X_n \backslash B_n$ is the identity map, so if we define $f_{n''}: S^k \to X_{n''}$ by $f_{n''}(s)=g(s)$ for each $s \in S^k$ then $f \simeq g = p_{nn''} f_{n''}$.

Case 1 ($k=1$). Both S^1 and X_n are polyhedra, so there is a simplicial map $h: S^1 \to X_n$ such that $h \simeq f$. The image of h is a compact 1-dimensional subpolyhedron of 2-dimensional polyhedron X_n, so there is a small disk D in $X_n \backslash h(S^1)$. It is clear that there is an isotopy $I_t: X_n \to X_n$ ($t \in [0,1]$) such that $I_0 = id_{X_n}$, and $I_1(D)=B_n$. So we may define g by $g=I_1 h$. Hence X is S^1-movable.

Case 2 ($k \geqq 2$). Since X_n is a closed orientable 2-manifold of genus n, it is well known ([8], p. 178) that the k-dimensional homotopy group of X_n is zero for $k=2,3,4,\cdots$ and for $n=1,2,3,\cdots$. Hence X is S^k-movable for $k \geqq 2$.

Propositon 2. *X is not 2-movable.*

Proof. Suppose that X is 2-movable. Then for each $n \in N$, there is $n' \in N$, $n' \geqq n$, such that for each $n'' \in N$, $n'' \geqq n$, there is a map $f_{n''}: X_{n'} \to X_{n''}$ satisfying that $p_{nn''} f_{n''} \simeq p_{nn'}$. (Because of $\dim X_{n'}=2$) Two maps $p_{nn'}$ and $p_{nn''}$ are both of degree 1 so $f_{n''}$ must be also of degree 1. But X_n is a closed orientable 2-manifold of genus n for $n \in N$, so by a theorem of H. Kneser [7] there is no map with degree 1 of $X_{n'}$ onto $X_{n''}$ if $n''>n'$. It contradicts our hypothesis.

Our proof of Proposition 2 is essentially the same as Borsuk's one in [2]. Also Proposition 2 follows from Borsuk [2] and Kodama-Watanabe [6 Theorem 3].

The author wishes to thank Professor Y. Kodama for his helpful comments.

References

[1] K. Borsuk: On movable compacta. Fund. Math., **66**, 137–146 (1969).
[2] ——: On a locally connected non-movable continuum. Bull. Acad. Polon. Sci. Ser. sci. math. astronom. et phys., **17**, 425–430 (1969).

[3] K. Borsuk: On the n-movability. ibid., **20**, 859–864 (1972).
[4] ——: On some hereditable shape properties. Ann. Polon. Math., **29**, 83–86 (1974).
[5] S. Mardešić and J. Segal: Movable compacta and ANR-systems. Bull. Acad. Polon., **18**, 649–654 (1970).
[6] Y. Kodama and T. Watanabe: A note on Borsuk's n-movability. ibid, **22**, 289–294 (1974).
[7] H. Kneser: Die kleinste Bedeckungszahl innerhalb einer Klasse von Flachenabbildungen. Math. Ann., **103**, 347–358 (1930).
[8] S. Lefschetz: Introduction to Topology. Princeton University Press.