# 14. *Propagation of Singularities of Solutions for Pseudodifferential Operators with Multiple Characteristics and their Local Solvability*

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In this note we consider propagation of singularities of solutions for pseudodifferential operators with multiple real characteristics and local solvability. It has become clearer from recent investigations that localization in $(x, \xi)$-space (micro-localization) is useful in order to study regularity and solvability. Our results stated in this note are obtained by micro-local considerations. Details will be published elsewhere.

Let $\Omega$ be a domain in $\boldsymbol{R}^n$. $(x, \xi)$ denotes a point in $\Omega \times (\boldsymbol{R}^n - \{0\})$ and $D = \frac{1}{\sqrt{-1}} \frac{\partial}{\partial x}$. $P(x, D) \in L^m(\Omega)$ means that $P(x, D)$ is a classical pseudodifferential operators of order $m$. Let $\Gamma$ be an open conic subset of $\Omega \times (\boldsymbol{R}^n - \{0\})$. We set $L^m(\Omega, \Gamma) = \{P(x, D) \in L^m(\Omega)\,;\, \text{supp}\, p(x, \xi) \subset \Gamma\}$, where $p(x, \xi)$ is the symbol of $P(x, D)$.

Definition. An operator $P(x, D) \in L^m(\Omega)$ has constant multiple real characteristics, if the principal symbol $p_m(x, \xi)$ is decomposed into

(1) $$p_m = (p^1)^{m_1}(p^2)^{m_2} \cdots (p^s)^{m_s},$$

where $p^i$ $(i = 1, 2, \cdots s)$ has the following properties:

Each $p^i$ is real valued, positively homogenous in $\xi$ and $\text{grad}_\xi\, p^i \neq 0$ on the characteristic manifold $\Sigma_{p^i} = \{(x, \xi)\,;\, p^i(x, \xi) = 0\}$ and if $j \neq k$, $\Sigma_{p^j} \cap \Sigma_{p^k} = \phi$.

Hereafter we assume that $P(x, D) \in L^m(\Omega)$ is properly supported and has constant multiple real characteristics. So in a conic neighbourhood $\Gamma$ of $(x_0, \xi_0) \in \Sigma_p = \{(x, \xi)\,;\, p_m(x, \xi) = 0\}$ we can factorize the principal symbol as follows:

(2) $$p_m(x, \xi) = a(x, \xi) q(x, \xi)^k,$$

where $q(x, \xi)$ is real valued, positively homogenous of order 1 in $\xi$, and $q(x_0, \xi_0) = 0$, $\text{grad}_\xi\, q(x_0, \xi_0) \neq 0$ and $a(x, \xi)$ is real valued, positively homogenous of order $m - k$ in $\xi$ and $a(x, \xi) \neq 0$ on $\Gamma$.

In order to micro-localize our study, we introduce a function $\varphi(x, \xi) \geqq 0$ positively homogenous of order 0 in $\xi$ such that $\varphi \equiv 1$ in a conic neighbourhood $\Gamma_1 (\Gamma_1 \subset \Gamma)$ of $(x_0, \xi_0)$ and $\varphi \equiv 0$ outside of $\Gamma$.

As for micro-local regularity, we have

**Theorem 1.** *Suppose that there are* $B^i_j(x, D) \in L^j(\Omega)$ *with the* *principal symbol* $b_j^i(x,\xi)$ *and* $G(x,D)\in L^0(\Omega,\Gamma_1)$ *whose symbol* $g(x,\xi)$ *equals to* 1 *on a conic neighbourhood* $\Gamma_2(\Gamma_2\subset\Gamma_1)$ *of* $(x_0,\xi_0)$ *such that one of the following conditions holds*:

(3) $G(x,D)P(x,D)\equiv G(x,D)A_\varphi(x,D)Q_\varphi(x,D)^k+B^1_{m-1}(x,D)$ $(\mathrm{mod}\, L^{m-2}(\Omega,\Gamma))$, $k\geqq 2$ *and* $\mathrm{Im}\, b^1_{m-1}(x,\xi)\neq 0$ *on* $\sum_q\cap\Gamma_2$,

(4) $G(x,D)P(x,D)\equiv G(x,D)A_\varphi(x,D)Q_\varphi(x,D)^k+B^2_{m-k}(x,D)Q_\varphi(x,D)^{k-1}+B^3_{m-2}(x,D)\ (\mathrm{mod}\, L^{m-3}(\Omega,\Gamma))$, $k\geqq 3$ *and* $\mathrm{Im}\, b^3_{m-2}(x,\xi)\neq 0$

*on* $\Sigma_q\cap\Gamma_2$, *where* $A_\varphi(x,D)(Q_\varphi(x,D))$ *is an operator with the symbol* $a(x,\xi)\varphi(x,\xi)$ (*resp.* $q(x,\xi)\varphi(x,\xi)$).

Then $(WF(u)-WF(f))\cap\Sigma_q=\phi$ for every solution $u\in\mathcal{D}'(\Omega)$ of $Pu=f$.

As for propagation of singularities, we have

**Theorem 2.** *Suppose that one of conditions holds*:

(5) $G(x,D)P(x,D)\equiv G(x,D)A_\varphi(x,D)Q_\varphi(x,D)^k+B^4_{m-2}(x,D)Q_\varphi(x,D)$ $(\mathrm{mod}\, L^{m-2}(\Omega,\Gamma))$, $k\geqq 3$ *and* $\mathrm{Im}\, b^4_{m-2}(x,\xi)\neq 0$ *on* $\Sigma_q\cap\Gamma_2$,

(6) $G(x,D)P(x,D)\equiv G(x,D)A_\varphi(x,D)Q_\varphi(x,D)^k+B^5_{m-3}(x,D)Q_\varphi(x,D)^2+B^6_{m-3}(x,D)Q_\varphi(x,D)\ (\mathrm{mod}\, L^{m-3}(\Omega,\Gamma))$, $k\geqq 4$ *and* $\mathrm{Im}\, b^5_{m-3}(x,\xi)\neq 0$ *on* $\Sigma_q\cap\Gamma_2$.

*Then* $(WF(u)-WF(f))\cap\Gamma$ *is contained in* $\Sigma_q$ *and invariant under the Hamiltonian vector field* $H_q$ *for every solution* $u\in\mathcal{D}'(\Omega)$ *of* $Pu=f$.

By making use of Leibnitz formula of pseudodifferential operators, we have

**Theorem 3.** *Let* $p_m=(p^1)^{m_1}(p^2)^{m_2}\cdots(p^s)^{m_s}$ *be as in Definition. Suppose that one of conditions* (7) *and* (8) *holds*:

*Set* $S=p_{m-1}-\dfrac{1}{2\sqrt{-1}}\sum_{i=1}^n\dfrac{\partial^2}{\partial x_i\partial\xi_i}p_m$,

(7) $m_k\geqq 2$, $\mathrm{Im}\, S\neq 0$ *on* $\Sigma_{pk}$,

(8) $m_k\geqq 3$, $S=dS=d^2S=\cdots=d^{k-1}S=0$ *and*

$$\mathrm{Im}\left(p_{m-2}-\frac{1}{2\sqrt{-1}}\sum_{i=1}^n\frac{\partial^2}{\partial x_i\partial\xi_i}p_{m-1}\right)\neq 0 \text{ on } \Sigma_{pk}.$$

*Then* $(WF(u)-WF(f))\cap\Sigma_{pk}=\phi$ *for every solution* $u\in\mathcal{D}'(\Omega)$ *of* $Pu=f$.

**Theorem 4.** *Suppose that one of the following conditions holds*:

(9) $m_k\geqq 3$, $S=0$ *and* $\mathrm{Im}\, dS\neq 0$ *on* $\Sigma_{pk}$,

(10) $m_k\geqq 4$, $S=dS=0$, $\mathrm{Im}\, d^2S\neq 0$ *and*

$$p_{m-2}-\frac{1}{2\sqrt{-1}}\sum_{i=1}^n\frac{\partial^2}{\partial x_i\partial\xi_i}p_{m-1}-\frac{1}{8}\sum_{i,j=1}^n\frac{\partial^4}{\partial x_i\partial\xi_i\partial x_j\partial\xi_j}p_m=0 \text{ on } \Sigma_{pk}.$$

*Then* $(WF(u)-WF(f))\cap\Sigma_{pk}$ *is invariant under the Hamiltonian vector field* $H_{pk}$ *for every solution* $u\in\mathcal{D}'(\Omega)$ *of* $Pu=f$.

Combining Theorem 3 and Theorem 4, we have

**Theorem 5.** *If one of conditions* (7), (8), (9), (10) *and* (11) *below holds on* $\Sigma_{pk}$:

(11) *characteristic manifold* $\Sigma_{pk}$ *satisfies the Levi's condition* ([1]).
*Then P is locally solvable.*

**Remark 1.** For operators with simple characteristics Duistermaat-Hörmander [2] and for operators satisfying the Levi's condition Chazarain [1] investigate propagation of singularities and existence of solutions. Matsumoto [3] proves local solvability under the conditions (7) and (9) by the method different from ours.

**Remark 2.** We can give other conditions under which singularities of solutions of operators propagate along the bicharacteristics or they are micro-local hypoelliptic in the form of division of pseudodifferential operators as in Theorem 1 and Theorem 2.

## References

[1] Chazarain, J.: Propagation des singularités pour une classe d'opérateurs a caractéristiques multiples et résolubilité locale. Ann. Inst. Fourier, **24**, 203–223 (1974).

[2] Duistermaat, J. J., and Hörmander, L.: Fourier integral operators. II. Acta Math., **128**, 183–269 (1972).

[3] Matsumoto, W.: Local solvability of class of partial differential operators with multiple characteristics. Proc. Japan Acad., **51**, 151–155 (1975).