31. A Local Study of Some Additive Problems in the Theory of Numbers

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In this note we shall give some supplemental remarks to the author's previous work [5] on Goldbach's problem. As an application we shall solve some analogues of Titchmarsh divisor problem. The details of the latter will appear elsewhere.

We start with stating the following theorem which plays an essential role in our argument and whose slightly weaker form is proved and used in [5].

Theorem 1. *Suppose that* $\sum_{m\leqslant x}|b(m)|^2 \ll x(\log x)^C$ *with some positive absolute constant C. Then for any positive constants A and b* (<1), *there exists a positive constant B such that*

$$\sum_{d\leqslant Q}\operatorname*{Max}_{(a,d)=1}\left|\sum_{\substack{1\leqslant m\leqslant x^\delta\\(m,d)=1}} b(m)\left(\sum_{\substack{p\leqslant x/m\\ p\equiv am^*(\mathrm{mod}\ d)}} 1-\frac{Li(x/m)}{\varphi(d)}\right)\right| \ll x(\log x)^{-A}$$

uniformly for δ *in* $0\leqslant\delta<1-(\log x)^{-b}$, *where* $Q=x^{1/2}(\log x)^{-B}$, $mm^*\equiv 1$ $(\mathrm{mod}\ d)$, $Li(x)=\int_2^x(\log x)^{-1}dx+0(1)$ *and p runs over primes.*

In fact, this is a generalization of Bombieri's mean value theorem (namely, $\delta=0$ and $b(1)=1$, Cf. [2] and [6]) and Chen's argument in [3]. We can also prove the following inequality under $b(m)\ll x^{1-\delta-\beta}$ for $m\leqslant x^\delta$, $\beta=(\log x)^{-f}$ with some f in $b<f<1$ in addition to the same circumstance as above;

$$\sum_{d\leqslant Q}\operatorname*{Max}_{(a,d)=1}\operatorname*{Max}_{1\leqslant y\leqslant x}\left|\sum_{\substack{mp\leq y, m\leqslant x^\delta\\ mp\equiv a(\mathrm{mod}\ d)}} b(m)-\frac{1}{\varphi(d)}\sum_{\substack{mp\leqslant y\\ m\leqslant x^\delta}} b(m)\right| \ll x(\log x)^{-A}.$$

We call this Theorem 1′. The conclusion in Theorem 1 (similarly for 1′) holds even if we replace $\sum_{\substack{p\leqslant x/m\\ p\equiv am^*(\mathrm{mod}\ d)}} 1-\frac{Li(x/m)}{\varphi(d)}$ by $\sum_{\substack{p\leqslant x^{1-\delta}\\ p\equiv am^*(\mathrm{mod}\ d)}} 1 -\frac{Lix^{1-\delta}}{\varphi(d)}$. We denote this by Theorem 1″. When our conclusion holds for any positive A and $Q=x^{1-\varepsilon}$ with any positive ε, we call it the generalized Halberstam-Richert's conjecture (G.H.R.).

Now let N be a sufficiently large even integer. Let $G(N)$ be the number of primes $p\leqslant N$ such that $N-p$ is a prime. Then

$$G(N)=P_N(N,N^{1/\alpha})-M_1(\alpha)-M_2(\alpha,2)+0(N^{1/\alpha})$$

for any α in $2<\alpha\leqslant 3$, where we put $P_N(N, N^{1/\alpha})=|\{p\leqslant N\,;\, p'\nmid N-p \text{ for any } p'\leqslant N^{1/\alpha}\}|$, $M_1(\alpha)=|\{p\leqslant N\,;\, N-p=p_1p_2,\ N^{1/\alpha}<p_1\leqslant p_2\leqslant N^{1/2}\}|$ and $M_2(\alpha,\beta)=|\{p\leqslant N\,;\, N-p=p_1p_2,\ N^{1/\alpha}<p_1\leqslant N^{1/\beta}<p_2\}|$ for $\alpha\geqslant\beta\geqslant 2$. For a subset I of $(1, N^{1/2}]$, let $G_2(N, I)$ be the number of primes $p\leqslant N$ such that $N-p$ is a prime or $N-p=p_1p_2$ with primes p_1 and p_2 satisfying $p_1\in I$ and $p_1\leqslant p_2$. Chen [3] has shown that

$$G_2(N,(1,N^{1/10}])>\frac{0.67NC_N}{(\log N)^2} \quad\text{with}\quad C_N=\prod_{\substack{p|N\\p>2}}\frac{p-1}{p-2}\prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right).$$

(Cf. also Notes in p. 338 of [7].) In [5] we have shown that

$$G_2(N,(1,N^{1/9.26}])>ANC_N(\log N)^{-2}$$

and that N is written as a sum of a prime and a natural number which has at most one prime factor less than $N^{1089/2089}$. To maximize $I\subset(1,N^{1/2}]$ such that $G_2(N,I)>ANC_N(\log N)^{-2}$, or to estimate $G(N)$ itself, we need estimates of P_N, M_1 and M_2 above. Using a weaker form of Theorem 1 we have shown in [5] that

$$M_2(\alpha,\beta)\leqslant 8(1+\varepsilon)\log((\alpha-1)/(\beta-1))NC_N(\log N)^{-2}.$$

If we use Theorem 1″, then we get

Theorem 2. *$M_1(\alpha)\leqslant(16+\varepsilon)NC_N(\log N)^{-3}$ for $\alpha>2$.*

If we assume (G.H.R.), then we get

$$4(1+\varepsilon)NC_N(\log N)^{-2}\geqslant P_N(N,N^{1/\alpha})>4(1-\varepsilon)\log(\alpha-1)NC_N(\log N)^{-2}$$

and

$$M_2(\alpha,\beta)\leqslant 4(1+\varepsilon)\log((\alpha-1)/(\beta-1))NC_N(\log N)^{-2}.$$

We have also

$$G(N)+M_2(\alpha,2)\geqslant 4(1-\varepsilon)\log(\alpha-1)NC_N(\log N)^{-2}-(8+\varepsilon)NC_N(\log N)^{-3}.$$

Hence, in particular, we get

Theorem 3 (On (H.R.)). *For any positive ε,*

$$G_2(N,(1,N^{112-\varepsilon}])>ANC_N(\log N)^{-2}.$$

So, (G.H.R.) almost solves Goldbach's problem, but it is not enough.

Now we turn to the analogues of Titchmarsh divisor problem. In [12] Titchmarsh posed and solved under the generalized Riemann Hypothesis, the problem of an asymptotic behavior of the sum $\sum_{p\leqslant x}\tau(p-1)$, where we put $\tau(m)=\sum_{d|m}1$. In [9] Linnik solved this unconditionally using his dispersion method (cf. also [4] and [11]). Here we are concerned with the asymptotic behavior of the sum $\sum_{\substack{p_1\leqslant x^\delta,\\ p_2\leqslant x^{1-\delta}}}\tau(p_1p_2-1)$, where p_1 and p_2 run over primes and $0<\delta\leqslant 1/2$. Linnik's dispersion method solves this for $0<\delta<1/6$. Barban [1] solved this for $\delta=1/2$ using Linnik's large sieve. (Cf. [1] and [9] for a historical account of this subject and references.) Now we can solve this problem completely. Namely, using Theorem 1″, we can show

Theorem 4. *For any δ in $0<\delta\leqslant 1/2$, we have*

$$\sum_{p_1 \leqslant x^{\delta}, p_2 \leqslant x^{1-\delta}} \tau(p_1 p_2 - 1) = \frac{315\zeta(3)}{2\pi^4 \delta(1-\delta)} \cdot \frac{x}{\log x} + 0(x(\log\log x)/(\log x)^2),$$

where $\zeta(s)$ is the Riemann zeta function.

If we use Theorem 1 for a constant δ in $0 < \delta < 1$, then we get

Theorem 5.

$$\sum_{p_1 p_2 \leqslant x} \tau(p_1 p_2 - 1) = (315/\pi^4)\zeta(3) x \log\log x + 0(x).$$

In a similar manner we can derive various asymptotic formulas for additive problems of numbers using Theorem 1, 1′ and 1″. For example, we can show without using the dispersion method an asymptotic formula for the number of the solutions of the equation $N = p_1 p_2 + x^2 + y^2$ for $p_1 p_2 \leqslant N$ (Cf. [9] and [10] for a proof using the dispersion method.)

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