30. A Note on Explosion of Branching Markov Processes with Extinction

By Michio SHIMURA
Department of Mathematics, University of Tsukuba

(Comm. by Kôsaku YOSIDA, M. J. A., March 12, 1976)

1. Preliminary. We discuss the explosion problem of branching Markov process under extinction effect. Such a problem was not considered in [3] and [4], since the existence of extinction brings some difficulty on the probabilistic consideration.[1] The difficulty will be removed through the auxiliary procedure which will be presented below.

Let S be a locally compact Hausdorff space with the second countability. Let $\boldsymbol{S}$ be the topological sum of the symmetric product spaces $S^{(n)}$, $n=0,1,\cdots,\infty$, with $S^{(0)}=\{\partial\}$ and $S^{(\infty)}=\{\Delta\}$. Let $\boldsymbol{X}=(\Omega,\boldsymbol{X}_t,\boldsymbol{P}_x)$ be a branching Markov process on the state space $\boldsymbol{S}$ in the sense of [1]. For $\boldsymbol{X}$ define the extinction time by $e_\partial=\inf\{t;\boldsymbol{X}_t=\partial\}$ and the explosion time by $e_\Delta=\inf\{t;\boldsymbol{X}_t=\Delta\}$.[2] Let $\{\boldsymbol{T}_t\}_{t\geqslant 0}$ be the semi-group of $\boldsymbol{X}$ acting on $C_0(\boldsymbol{S})$.[3] Set $q(x)=\lim_{t\to\infty}\boldsymbol{T}_t\hat{0}(x)=\boldsymbol{P}_x(e_\partial<\infty)$ for $x\in S$, where for each function f on S a function $\hat{f}$ on $\boldsymbol{S}$ is defined as follows; $\hat{f}(\partial)=1$, $\hat{f}(\Delta)=0$ and $\hat{f}(\boldsymbol{x})=f(x_1)\cdots f(x_n)$ if $\boldsymbol{x}=[x_1,\cdots,x_n]\in S^{(n)}$, $n=1,2,\cdots$. Throughout this article we assume

(*Asm.*) *$q(x)$ is a continuous function on S such that $0\leqslant q(x)<1$, $x\in S$.*

Let us define the family of operators $\{\tilde{\boldsymbol{T}}_t\}_{t\geqslant 0}$ for $\hat{f}\in C_0(\boldsymbol{S})$ with a continuous function f on S such that $0\leqslant f(x)<1$ for $x\in S$.

$$(1)\qquad \tilde{\boldsymbol{T}}_t\hat{f}(x)=\frac{1}{1-q(x)}\{\boldsymbol{T}_t(\widehat{q+(1-q)f})(x)-q(x)\},\qquad x\in S.$$

Following [1] $\{\tilde{\boldsymbol{T}}_t\}_{t\geqslant 0}$ is uniquely extended to a branching semi-group acting on $C_0(\boldsymbol{S})$, and we also denote the extension by $\{\tilde{\boldsymbol{T}}_t\}_{t\geqslant 0}$. $\{\tilde{\boldsymbol{T}}_t\}_{t\geqslant 0}$ determines a branching Markov process $\tilde{\boldsymbol{X}}$ on $\boldsymbol{S}$ (cf. [1]). We call the process $\tilde{\boldsymbol{X}}$ the associated (branching Markov) process to $\boldsymbol{X}$.

2. Results and the proof.

Lemma 1. *Let $\tilde{\boldsymbol{X}}$ be the associated process to $\boldsymbol{X}$, then*

(i) *$\boldsymbol{X}$ is explosive if and only if $\tilde{\boldsymbol{X}}$ is explosive.*

(ii) *If $\tilde{\boldsymbol{X}}$ is explosive with probability one, then*

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1) For the terminologies used in our note, refer [3] and [4].

2) We define $\inf\{\emptyset\}=\infty$.

3) $C_0(\boldsymbol{S})=\{f;$ continuous function on $\boldsymbol{S}$ which vanishes at the infinities of $\boldsymbol{S}\}$, where the infinities consist of Δ and the infinity of the one point compactification of $S^{(n)}$, $n=1,2,\cdots$.

$$P_x(e_\partial<\infty \text{ or } e_\Delta<\infty)=1 \qquad \text{for } x\in S.$$

Proof. (i) Setting $f=1$ in (1), and letting $t\to\infty$, we have

$$(2) \qquad \tilde{P}_x(\tilde{e}_\Delta=\infty)=\frac{1}{1-q(x)}\{P_x(e_\Delta=\infty)-q(x)\}, \qquad x\in S.^{4)}$$

(i) follows from (2).

(ii) Since $\tilde{P}_x(\tilde{e}_\Delta=\infty)=0$ for $x\in S$, we have from (2) $P_x(e_\Delta=\infty)=q(x)$ for $x\in S$. Hence we have $P_x(e_\partial<\infty)+P_x(e_\Delta<\infty)=1$ for $x\in S$. Noting $\{e_\partial<\infty\}\cap\{e_\Delta<\infty\}=\emptyset$, we obtain (ii).

Lemma 2. *Let X be the (X,k,π)-branching Markov process,*[5] *where the branching law is given by*

$$(3) \qquad \pi(x,dy)=p_0\delta_\partial(dy)+\sum_{n=2}^{\infty}p_n\delta_{[\underbrace{x,\cdots,x}_{n}]}(dy), \qquad x\in S,\ dy\subset \boldsymbol{S},^{6)}$$

and p_n, $n=0,2,3,\cdots$, is a probability sequence. Suppose that $P_x(e_\partial<\infty)=q$ for $x\in S$ for some constant $q\in(0,1)$. Then the associated process $\tilde{X}$ is the $(X,\tilde{k},\tilde{\pi})$-branching Markov process, where

$$\tilde{k}(x)=(1-\tilde{p}_1)k(x) \quad \textit{and} \quad \tilde{\pi}(x,dy)=\sum_{n=2}^{\infty}\frac{\tilde{p}_n}{1-\tilde{p}_1}\delta_{[\underbrace{x,\cdots,x}_{n}]}(dy),$$
$$x\in S,\ dy\subset\boldsymbol{S},$$

where $\tilde{p}_n$, $n=1,2,\cdots$, is the probability sequence determined by the following identity

$$(4) \qquad \frac{1}{1-q}\left\{\sum_{n=2}^{\infty}p_n(q+(1-q)\xi)^n+p_0-q\right\}=\sum_{n=1}^{\infty}\tilde{p}_m\xi^n, \qquad 0\leqslant\xi\leqslant 1.$$

Proof. Here we are content with the following heuristic proof. Let f be a suitably "smooth" function on S with $0\leqslant f(x)<1$ for $x\in S$, then we have from [1; III]

$$(5) \qquad \lim_{t\to 0}\frac{T_t\hat{f}(x)-f(x)}{t}=\mathcal{G}f(x)+k(x)\left\{p_0+\sum_{n=2}^{\infty}p_nf(x)^n-f(x)\right\},$$

where $\mathcal{G}$ is the infinitesimal operator of X. By the relation (1) between $\{T_t\}_{t\geqslant 0}$ and $\{\tilde{T}_t\}_{t\geqslant 0}$, and by (5)

$$\lim_{t\to 0}\frac{\tilde{T}_t\hat{f}(x)-f(x)}{t}$$

$$(6) \qquad =\frac{1}{1-q}\lim_{t\to 0}\frac{T_t\widehat{(q+(1-q)f)}(x)-(q+(1-q)f(x))}{t}$$

$$=\mathcal{G}f(x)+k(x)\left(\frac{1}{1-q}\sum_{n=2}^{\infty}p_n(q+(1-q)f(x))^n+p_0-q-f(x)\right).$$

Let us rewrite the last term of (6) using (4), then we have

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4) $\tilde{P}_\cdot(*)$, $\tilde{e}_\partial$ and $\tilde{e}_\Delta$ are the probability measure, extinction time and explosion time of $\tilde{X}$, respectively.

5) For the definition of (X,k,π)-branching Markov process, refer [3].

6) $\delta_x(\cdot)$ is a measure on $\boldsymbol{S}$ having a unit mass only on x.

$$\lim_{t\to 0}\frac{\tilde{T}_t\hat{f}(x)-f(x)}{t}=\mathcal{G}f(x)+\tilde{k}(x)\left\{\sum_{n=2}^{\infty}\frac{\tilde{p}_n}{1-\tilde{p}_1}f(x)^n-f(x)\right\}\qquad x\in S.$$

Hence $\boldsymbol{X}$ is the $(X,\tilde{k},\tilde{\pi})$-branching Markov process (cf. [1]).

Lemma 3. *Let $\boldsymbol{X}$ be the same as in Lemma 2. Suppose that $\boldsymbol{P}_x(\tau<\infty)=1$ for $x\in S$, where τ is the first branching time defined by $\tau=\inf\{t;\boldsymbol{X}_t\notin S^{(n)}\}$ if $\boldsymbol{X}_0\in S^{(n)}$. Then $\boldsymbol{P}_x(e_\partial<\infty)=q$ for $x\in S$, where q is the minimal root of the equation*

$$\xi=F(\xi)=p_0+\sum_{n=2}^{\infty}p_n\xi^n,\qquad \xi\in[0,1].$$

Our proof is essentially owe to [2] (refer also [5]), so we omit it.

Proposition. *Let $X=(W,X_t,P_x)$ be a temporally homogeneous Lévy process on the real line R satisfying the condition $P_0(\sup_{0\leqslant t<\infty}X_t=\infty)=1$. Let $k(x)$ be a non-negative continuous function on R satisfying $\lim_{x\to\infty}k(x)=\infty$. Let $\pi(x,d\boldsymbol{y})=p\delta_\partial(d\boldsymbol{y})+(1-p)\delta_{[x,x]}(d\boldsymbol{y})$ $(x\in R, d\boldsymbol{y}\subset\boldsymbol{R})$ for any constant $p\in(0,1/2)$.*[7] *Consider the (X,k,π)-branching Lévy process $\boldsymbol{X}$. Then*

(i) *$q(x)=\boldsymbol{P}_x(e_\partial<\infty)=p/(1-p)$ for $x\in R$.*

(ii) *The associated process $\tilde{\boldsymbol{X}}$ is the $(X,(1-2p)k(x),\delta_{[x,x]}(d\boldsymbol{y}))$- branching Lévy process.*

(iii) *If $\tilde{\boldsymbol{X}}$ is explosive with probability one, then $\boldsymbol{X}$ is also explosive, not with probability one but $\boldsymbol{P}_x(e_\Delta<\infty)=\boldsymbol{P}_x(e_\partial=\infty)=(1-2p)/(1-p)$ for $x\in R$.*

Proof. To prove (i) check $\boldsymbol{P}_x(\tau<\infty)=1$ for $x\in R$, which follows from

$$\boldsymbol{P}_x(\tau=\infty)=\lim_{t\to\infty}\boldsymbol{P}_x(\tau>t)=\lim_{t\to\infty}\int_W\exp\left\{-\int_0^t k(X_s)\,\mathrm{d}s\right\}dP_x=0.$$

by the condition given to X and $k(x)$. Then by Lemma 3 we obtain $q(x)\equiv p/(1-p)$. (ii) is a direct consequence of (i) and Lemma 2. (iii) is a direct consequence of (i) and Lemma 1.

Remark. By the condition given to X and $k(x)$ and by the branching law, Proposition 1 or 2 in [4] is well applicable on $\tilde{\boldsymbol{X}}$ to obtain a sufficient condition for the explosion with probability one.

References

[1] Ikeda, N., Nagasawa, M., and Watanabe, S.: Branching Markov processes I, II and III. J. Math. Kyoto Univ., **8**(2), 233–278, **8**(3), 365–410 (1968), **9**(1), 95–160 (1969).

[2] Sevast'yanov, B. A.: Branching stochastic processes for particles diffusing in a bounded domain with absorbing boundaries (English transl.). Theo. Prob. Appl., **3**(2), 111–126 (1958).

[3] Shimura, M.: The explosion problem of branching stable processes. J. Math. Kyoto Univ., **14**(1), 229–253 (1974).

7) When $p\in[1/2,1]$, we have from Lemma 3 $q(x)\equiv 1$, and our (Asm.) fails.

[4] Shimura, M.: The explosion problem of branching Lévy processes (to appear in J. Math. Kyoto Univ.).
[5] Watanabe, S: On the branching process for Brownian particles with an absorbing boundary. J. Math. Kyoto Univ., **4**(2), 385–398 (1965).