27. *A Remark on the Character Rings of Finite Groups*

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Introduction. The integral group ring ZG of a finite abelian group G is an important example of Gorenstein ring of dimension one (see [1], [2]). In this case, since ZG is isomorphic to the character ring R_G of G, we say that R_G is a Gorenstein ring. In this paper we show that the character rings of arbitrary finite groups are Gorenstein rings.

1. Let G be a finite group. Then the character ring R_G of G is a commutative ring and a finitely generated free Z-module. Its unity element is the principal character of G. As for group rings ([3]), we see that *R_G is isomorphic to the dual* $\mathrm{Hom}_Z(R_G, Z)$ *as R_G-modules.* This is equivalent to the existence of a nondegenerate symmetric bilinear form $(\ ,\ ): R_G \times R_G \to Z$ which satisfies the following conditions:

1) $(rs, t) = (r, st)$ for $r, s, t \in R_G$.

2) For each $f \in \mathrm{Hom}_Z(R_G, Z)$, there exists an $s \in R_G$ such that $f(r) = (r, s)$ for $r \in R_G$.

Such a bilinear form $(\ ,\ )$ is given by

$$(r, s) = \langle \bar{r}, s \rangle$$

for $r, s \in R_G$, using the ordinary inner product

$$\langle \mu, \nu \rangle = \frac{1}{|G|} \sum_{x \in G} \bar{\mu}(x)\nu(x),$$

where $\bar{\mu}$ denotes the function defined by $\bar{\mu}(x) = \mu(x^{-1})$ for $x \in G$. In fact, if $(r, s) = 0$ for all $r \in R_G$, then $\langle \chi, s \rangle = 0$ for all irreducible characters χ of G. Hence $s = 0$, which shows that $(\ ,\ )$ is nondegenerate. Moreover, for each $f \in \mathrm{Hom}_Z(R_G, Z)$, put

$$s = \sum_{\chi} f(\bar{\chi})\chi,$$

where the sum is taken over all χ. Then $f(\chi) = (\chi, s)$ for all χ. Since $\{\chi\}$ is a Z-basis of R_G, we have $f(r) = (r, s)$ for all $r \in R_G$.

Hence R_G is a Frobenius Z-algebra in the sense of the definition given in [3]. It follows from Corollary 8 of [3] that R_G has a finite injective dimension. Thus from the fundamental theorem of [2] we obtain

Theorem 1. *The character rings of finite groups are Gorenstein rings.*

Let Λ be a commutative ring. Since the isomorphism $R_G \to \mathrm{Hom}_Z(R_G, Z)$ is extended to the isomorphism $\Lambda \otimes_Z R_G \to \mathrm{Hom}_\Lambda(\Lambda_Z \otimes_z R_G, \Lambda)$, we see that $\Lambda \otimes_Z R_G$ is a Frobenius Λ-algebra. This ring is a supplemented algebra under the mapping $\Lambda \otimes_Z R_G \to \Lambda$ given by $r \mapsto r(1)$ for $r \in \Lambda \otimes_Z R_G$. Therefore we have

$$\text{inj dim } \Lambda \otimes_Z R_G = \text{inj dim } \Lambda$$

(see Corollary 8′ of [3]). This yields

Corollary 1. *For a commutative ring Λ, the ring $\Lambda \otimes_Z R_G$ is Gorenstein if and only if Λ is Gorenstein.*

2. The next result gives us an example of a local Gorenstein ring of dimension one.

Let G be a finite p-group, and let $S = Z - pZ$. By Corollary 1 we see that $S^{-1}R_G$ is a Gorenstein ring. We shall prove that $S^{-1}R_G$ has only one maximal ideal $S^{-1}M_0$, where

$$M_0 = \{r \in R_G \mid r(1) \in pZ\}$$

is a maximal ideal of R_G.

It is evident that any maximal ideal of $S^{-1}R_G$ is of the form $S^{-1}M$ for some maximal ideal M of R_G such that $M \cap Z = pZ$. Let A be a Z-algebra generated by all $|G|$-th roots of 1. Then every maximal ideal M of R_G is expressible as

$$M = \{r \in R_G \mid r(c) \in \mathfrak{p}\}$$

for some $c \in G$ and some maximal ideal $\mathfrak{p}$ of A ([7]). Moreover we see that $M \cap Z = pZ$ implies $\mathfrak{p} \cap Z = pZ$. Since G is p-group, the unity element is only one p-regular element of G. Therefore, if $\mathfrak{p} \cap Z = pZ$, then $r(c) \equiv r(1) \pmod{\mathfrak{p}}$ for $r \in R_G$ and $c \in G$ (cf. Lemma 7 of § 10.3 in [7]). This shows that if $M \cap Z = pZ$, then $M = M_0$. Hence $S^{-1}M_0$ is a unique maximal ideal of $S^{-1}R_G$.

3. Finally we shall prove a result related to the above example. It is easily seen that the ring $S^{-1}R_G$ is a finitely generated $S^{-1}Z$-module and has no non-zero nilpotents.

Theorem 2. *Let A be a local ring of dimension one which satisfies the following conditions:*

1) *A has no non-zero nilpotents.*

2) *There exists a Dedekind subring R of A such that A is a finitely generated R-module. Then A is a Frobenius R-algebra if and only if A is a Gorenstein ring.*

It suffices to prove the "if" part. We need the following lemmas.

Let A be a Noetherian ring with unity element, and K the total quotient ring of A. For fractional ideals $\mathfrak{b}$ and $\mathfrak{a}$ in K, let $\mathfrak{b} : \mathfrak{a}$ denote the set of all elements x of K such that $x\mathfrak{a} \subseteq \mathfrak{p}$.

Lemma 1 ([4], Lemma 2.1). *Let $\mathfrak{a}$ and $\mathfrak{b}$ be fractional ideals in K such that $\mathfrak{a}K=K=\mathfrak{b}K$. Then there exists an isomorphism $\alpha: \mathfrak{a}:\mathfrak{b} \to \mathrm{Hom}_A(\mathfrak{b},\mathfrak{a})$ as A-modules which maps each x of $\mathfrak{a}:\mathfrak{b}$ to the multiplication by x.*

Lemma 2 ([4], Lemma 2.3). *Let the notation be as above. Then there exists an isomorphism $\mathfrak{a}: (\mathfrak{a}:\mathfrak{b}) \to \mathrm{Hom}_A(\mathrm{Hom}_A(\mathfrak{b},\mathfrak{a}),\mathfrak{a})$ such that the diagram*

$$\begin{array}{ccc} \mathfrak{b} & \xrightarrow{i} & \mathfrak{a}:(\mathfrak{a}:\mathfrak{b}) \\ & {}_{c}\searrow & \downarrow \\ & & \mathrm{Hom}_A(\mathrm{Hom}_A(\mathfrak{b},\mathfrak{a}),\mathfrak{a}) \end{array}$$

is commutative, where i is the inclusion and c is the canonical mapping.

Corollary. *Let the notation be as above. If A is a Gorenstein ring, then we have $\mathfrak{b}=A:(A:\mathfrak{b})$.*

Proof. From Lemma 2 it follows that the mapping $c: \mathfrak{b} \to \mathrm{Hom}_A(\mathrm{Hom}_A(\mathfrak{b},A),A)$ is injective (i.e. $\mathfrak{b}$ is torsionless). By Theorem (6.2) of [2] we see that c is an isomorphism, hence $\mathfrak{b}=A:(A:\mathfrak{b})$.

We turn to the proof of the "if" part of Theorem 2. We assume that A is a local Gorenstein ring of dimension one and satisfies the conditions of Theorem 2. Let $S=R-\{0\}$. Then $S^{-1}R$ is the quotient field k of R. By hypothesis $K=S^{-1}A$ is semisimple and finitely generated as a k-module (moreover K is the total quotient ring of A). From Proposition 5 of [3] it follows that K is a Frobenius k-algebra, that is, K is isomorphic to $\mathrm{Hom}_k(K,k)$ as K-modules. We denote by A^* the image of $\mathrm{Hom}_R(A,R)$ under the embedding $\mathrm{Hom}_R(A,R) \to \mathrm{Hom}_k(K,k) \to K$. Then it is easily verified that $A^*K=K$. We shall prove that A^* is isomorphic to A, which completes the proof.

First we show that $A=A^*:A^*$. By Lemma 1 the following diagram is commutative:

$$\begin{array}{ccc} A & \xrightarrow{i} & A^*:A^* \\ & & \downarrow{\scriptstyle\alpha} \\ & & \mathrm{Hom}_A(A^*,A^*) \\ & {}_{c}\searrow & \downarrow{\scriptstyle\beta} \\ & & \mathrm{Hom}_A(\mathrm{Hom}_R(A,R),\mathrm{Hom}_R(A,R)) \\ & & \downarrow{\scriptstyle\gamma} \\ & & \mathrm{Hom}_R(\mathrm{Hom}_R(A,R),R) \end{array}$$

Here β is the mapping induced by the isomorphism $A^* \to \mathrm{Hom}_R(A,R)$ as A-modules, and γ is the natural isomorphism. Since A is a finitely generated torsion-free R-module and R is a Dedekind ring, we see that

A is a projective R-module. Therefore the canonical mapping c is an isomorphism. This implies $A=A^*:A^*$.

We have seen that $A^*K=K$, hence A^* has non-zero divisors of A. Since A^* is finitely generated as an A-module, there exists a non-zero divisor u of A such that uA^* is an ideal of A. From Theorem 124 of [5] it follows that uA^* is generated by non-zero divisors of A, say $u_1, \cdots, u_r$. Put $v_i=u_i^{-1}u$. Then we have $A=\bigcap_{i=1}^r v_iA^*$. Indeed, $\bigcap v_iA^*=A^*:\sum Av_i^{-1}=A^*:A^*=A$. By Corollary of Lemma 2 we have

$$\sum_i (A:v_iA^*)=A:(A:\sum_i (A:v_iA^*)),$$

and

$$A:\sum_i (A:v_iA^*)=\bigcap_i (A:(A:v_iA^*))=\bigcap_i v_iA^*=A,$$

hence $A=\sum_i (A:v_iA^*)$.

On the one hand, since A is a local ring and $A:v_iA^*$ are ideals of A, we see that $A=A:v_iA^*$ for some i. This implies $A^*=Av_i^{-1}$. Thus A^* is isomorphic to A, which proves our assertion.

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