26. *On the Irreducible Characters of the Finite Unitary Groups*

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Let k be a finite field, and k_2 the quadratic extension of k. The purpose of the present paper is to announce a theorem which gives a method to construct the irreducible characters of the finite unitary group $U_n(k_2)$ using those of the finite general linear group $GL_n(k_2)$, at least if the characteristic of k is not 2. As an application, we also obtain a parametrization of the irreducible characters of $U_n(k_2)$ which is dual to a known parametrization of the conjugacy classes. Proofs are omitted and will appear elsewhere.

1. Let $\mathfrak{G}$ be the general linear group $GL_n(K)$ over an algebraically closed field K of positive characteristic p. Let k be a finite subfield of K, and $k_m(\subset K)$ the extension of k of degree $m<\infty$. We denote by τ the Frobenius automorphism of K with respect to k. Then τ acts naturally on $\mathfrak{G}$ as an automorphism. Let σ be the automorphism of $\mathfrak{G}$ defined by

$$x^\sigma = (({}^t x)^\tau)^{-1} \qquad (x \in \mathfrak{G}),$$

where ${}^t x$ is the transposed matrix of $x \in \mathfrak{G}$. For a positive integer m, put

$$\mathfrak{G}_{\sigma^m} = \{x \in G \mid x^{\sigma^m} = x\}.$$

Then we have

$$\mathfrak{G}_{\sigma^m} = \begin{cases} GL_n(k_m) & \text{if } m \text{ is even,} \\ U_n(k_{2m}) & \text{if } m \text{ is odd.} \end{cases}$$

In the following, we fix m and put $G = \mathfrak{G}_{\sigma^m}$ and $G_\sigma = \mathfrak{G}_\sigma = U_n(k_2)$. The restriction of σ to G is an automorphism of the finite group G. In the following, we denote this automorphism also by σ. Let A be the cyclic group of order m generated by the automorphism σ of G. Assume that G and A are embedded in their semi-direct product GA. The following lemma is well known.

Lemma 1. *Let H be a finite group, and A a finite cyclic group generated by an automorphism σ of H. If an irreducible complex character χ of H is fixed by σ (i.e. satisfies $\chi(x^\sigma) = \chi(x)$ for all $x \in H$), then there exists an irreducible character $\tilde{\chi}$ of the semi-direct product HA whose restriction to H equals χ.*

For $x \in G = \mathfrak{G}_{\sigma^m}$, put $N(x) = x x^\sigma x^{\sigma^2} \cdots x^{\sigma^{m-1}}$.

Lemma 2. (a) *Let x be an element of G. The conjugacy class $C_G(N(x))$ of $N(x)$ in G contains an element of G_σ. Moreover, $C_G(N(x)) \cap G_\sigma$ forms a single conjugacy class of G_σ.*

(b) *Let x and y be elements of G such that the elements $x\sigma$ and $y\sigma$ of GA are conjugate to each other. Then, $C_G(N(x)) \cap G_\sigma = C_G(N(y)) \cap G_\sigma$.*

(c) *For $x \in G$, we denote by $C_{GA}(x\sigma)$ the conjugacy class of $x\sigma$ in GA. The correspondence $\mathfrak{N}$ from the set of GA-conjugacy classes of $\{x\sigma \mid x \in G\}$ into the set of conjugacy classes of G_σ defined by*

$$\mathfrak{N}(C_{GA}(x\sigma)) = C_G(N(x)) \cap G_\sigma \qquad (x \in G)$$

is bijective.

(d) $|C_{GA}(x\sigma)|\,|G|^{-1} = |C_G(N(x)) \cap G_\sigma|\,|G_\sigma|^{-1}$ *for all $x \in G$. (For a set S, $|S|$ denotes the number of its elements.)*

2. Theorem. *Assume that m is not divisible by p. Let χ be a σ-invariant irreducible character of G, and $\tilde{\chi}$ an extension of χ to an irreducible character of GA (see Lemma 1). Then there exists a unique irreducible character ψ_χ of G_σ which depends only on χ and satisfies*

$$\tilde{\chi}(x\sigma) = \pm \zeta^a \psi_\chi(n(x)) \qquad (x \in G),$$

where $n(x)$ is an arbitrary element of $C_G(N(x)) \cap G_\sigma$ (see Lemma 2), $\zeta = \exp(2\pi i/m)$, and a is an integer. Moreover, the mapping $\chi \to \psi_\chi$ is a bijection between the set of σ-invariant irreducible characters of G and the set of irreducible characters of G_σ. In particular, if char $(k) \neq 2$, *all the irreducible characters of G_σ may be obtained in this way.*

Remark 1. This theorem, and its proof, are valid even if one replace σ with the Frobenius automorphism τ. Using Green's deep results [2], Shintani [3] proved the τ-case without assuming that m is not divisible by p. Our proof is independent of Green's results [2].

3. Put $L = k_{2(n!)}$. We consider that σ acts on $L^\times = GL_1(k_{2(n!)})$ and on $\hat{L}^\times = \mathrm{Hom}(L^\times, \boldsymbol{C}^\times)$ by

$$t^\sigma = t^{-q}, \qquad u^\sigma(t) = u(t^{-q}) \qquad (t \in L^\times, u \in \hat{L}^\times),$$

where q is the number of elements of k. We denote by $\mathfrak{F}$ and $\hat{\mathfrak{F}}$ respectively, the set of σ-orbits in $L^\times$ and $\hat{L}^\times$. For an element f in $\mathfrak{F}$ (or $\hat{\mathfrak{F}}$) we denote by $d(f)$ the cardinality of the orbit f. Let $\mathfrak{P}$ be the set of partitions, i.e. decreasing sequences $\nu = (\nu_1, \nu_2, \cdots, \nu_r)$ of positive integers ν_i. For convention, we consider that $\mathfrak{P}$ contains the empty partition ϕ. For $\nu \in \mathfrak{P}$, put $|\nu| = \sum_i \nu_i$ if $\nu \neq \phi$, and $|\phi| = 0$. Using the Theorem with $m=2$ and a parametrization of the irreducible characters of $GL_n(k_2)$ due to J. A. Green [2], we see that the irreducible characters of $U_n(k_2)$ (char$(k) \neq 2$) are naturally parametrized by the set of functions $\hat{\lambda}: \hat{\mathfrak{F}} \to \mathfrak{P}$, which satisfies

$$\sum_{f \in \hat{\mathfrak{F}}} |\hat{\lambda}(f)|\, d(f) = n.$$

Remark 2. It is known [1] and easy to see that the conjugacy classes of $U_n(k_2)$ are naturally parametrized by the set of functions $\lambda: \mathcal{F} \to \mathcal{P}$, which satisfies

$$\sum_{f \in \mathcal{F}} |\lambda(f)|\, d(f) = n.$$

References

[1] V. Ennola: On the characters of the finite unitary groups. Ann. Acad. Scient. Fenn. A. I, No., **323** (1963).

[2] J. A. Green: The characters of the finite general linear groups. Trans. Amer. Math. Soc., **80**, 402–447 (1955).

[3] T. Shintani: Two remarks on irreducible characters of finite general linear groups (to appear in Jour. Math. Soc. Japan.).