49. Some Results on Additive Number Theory. I

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In his previous papers [2]–[5], the author gave some generalizations of the theorem of Erdös and Kac in [1]. In this note we shall give some theorems on additive number theory which are obtainable by similar methods as in the above papers. The detailed proofs will be given elsewhere.

Let k be an integer >1; let l_i $(i=1, \cdots, k)$ be positive integers, and put $l_0=l_1+\cdots+l_k$.

Theorem 1. *Let P_{ij} $(i=1, \cdots, k;\ j=1, \cdots, l_i)$ be sets, each consisting of prime numbers, subject to the following conditions:*

(C_1) *For each $i=1, \cdots, k$, the sets P_{ij} $(j=1, \cdots, l_i)$ are pairwise disjoint;*

(C_2) *As $x\to\infty$,*

$$\sum_{p\leq x, p\in P_{ij}} \frac{1}{p} = \lambda_{ij} \log\log x + o(\sqrt{\log\log x})$$

with positive constants λ_{ij} for $i=1, \cdots, k;\ j=1, \cdots, l_i$. (The sets P_{ij} with distinct i's need not be disjoint, and $P_{i1}\cup\cdots\cup P_{il_i}$ may not contain all primes.)

For a positive integer n, we denote by $\omega_{ij}(n)$ the number of distinct prime factors of n belonging to the set P_{ij}.

Let E be a Jordan-measurable set, bounded or unbounded, in the Euclidean space R^{l_0} of l_0 dimensions. For sufficiently large integer N, let $A(N; E)$ denote the number of representations of N as the sum of k positive integers: $N=n_1+\cdots+n_k$ such that the point $(x_{11}, \cdots, x_{1l_1}, \cdots, x_{k1}, \cdots, x_{kl_k})$ belongs to the set E, where

$$x_{ij} = \frac{\omega_{ij}(n_i) - \lambda_{ij}\log\log N}{\sqrt{\lambda_{ij}\log\log N}} \tag{1}$$

for $i=1, \cdots, k;\ j=1, \cdots, l_i$. Then, as $N\to\infty$, we have

$$A(N; E) \sim \frac{N^{k-1}}{(k-1)!}(2\pi)^{-(l_0/2)}\int_E \exp\left(-\frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{l_i} x_{ij}^2\right)dx_{11}\cdots dx_{kl_k}. \tag{2}$$

Theorem 2. *Let the polynomials $f_{ij}(\xi)$ $(i=1, \cdots, k;\ j=1, \cdots, l_i)$ of positive degree be subject to the following conditions:*

(C_1) *Each $f_{ij}(\xi)$ has rational integral coefficients, the leading coefficient being positive;*

(C_2) *Each $f_{ij}(\xi)$ is irreducible;*

(C_3) *For each $i, f_{ij}(\xi)$ ($j=1, \cdots, l_i$) are relatively prime in pairs.*

$\omega(n)$ will denote, for a positive integer n, the number of all distinct prime factors of n.

Let E be a Jordan-measurable set, bounded or unbounded, in the Euclidean space R^{l_0} of l_0 dimensions. For sufficiently large positive integer N, let $A(N; E)$ denote the number of representations of N as the sum of k positive integers: $N=n_1+\cdots+n_k$ such that $f_{ij}(n_i)>0$ and the point $(x_{11}, \cdots, x_{1l_1}, \cdots, x_{k1}, \cdots, x_{kl_k})$ belongs to the set E, where

$$x_{ij}=\frac{\omega\{f_{ij}(n_i)\}-\log\log N}{\sqrt{\log\log N}} \tag{3}$$

for $i=1, \cdots, k; j=1, \cdots, l_i$. Then, as $N\to\infty$, we have again the same formula as (2).

We could restate this theorem on removing the condition (C_2), but then the enunciation would become more complicated. We could also state a theorem which would contain Theorems 1 and 2 as special cases.

The statement of the Theorem 1 remains true, when we replace $\omega_{ij}(n_i)$ in (1) by $\Omega_{ij}(n_i)$, the number of prime factors of n_i belonging to the set P_{ij}, multiple factors being counted multiply, or when we replace $\omega_{ij}(n_i)$ by $\log\tau_{ij}(n_i)/\log 2$, where $\tau_{ij}(n_i)$ stands for the number of positive divisors of n_i which are composed only of primes belonging to the set P_{ij}.

Also, the statement of the Theorem 2 remains true, when we replace $\omega\{f_{ij}(n_i)\}$ in (3) by $\Omega\{f_{ij}(n_i)\}$, the number of all prime factors of $f_{ij}(n_i)$, multiple factors being counted multiply, or when we replace $\omega\{f_{ij}(n_i)\}$ by $\log\tau\{f_{ij}(n_i)\}/\log 2$, where $\tau\{f_{ij}(n_i)\}$ stands for the number of all positive divisors of $f_{ij}(n_i)$.

We mention now some special cases of Theorems 1 and 2 which might be of interest.

Theorem 3. *Let $\alpha_i<\beta_i$ ($i=1, \cdots, k$). For sufficiently large N, let $A(N)=A(N; \alpha_1, \beta_1, \cdots, \alpha_k, \beta_k)$ denote the number of representations of N as the sum of k positive integers: $N=n_1+\cdots+n_k$ such that the inequalities*

$$\log\log N+\alpha_i\sqrt{\log\log N}<\omega(n_i)<\log\log N+\beta_i\sqrt{\log\log N} \tag{4}$$

hold for $i=1, \cdots, k$ simultaneously. Then, as $N\to\infty$, we have

$$A(N)\sim\frac{N^{k-1}}{(k-1)!}(2\pi)^{-k/2}\prod_{i=1}^{k}\int_{\alpha_i}^{\beta_i}e^{-x^2/2}dx.$$

Theorem 4. *The statement of the Theorem 3 remains true when we replace $\omega(n_i)$ in (4) by $\omega(n_i+1)$.*

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