59. *Invariant Measures for Bounded Amenable Semigroups of Operators*

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In this paper we consider the invariant measure problem for bounded amenable semigroups of positive L_1-operators. A necessary and sufficient condition is given for the existence of finite equivalent invariant measures for such semigroups.

Let $(X, \mathcal{F}, m)$ be a probability space and let $L_p(X)=L_p(X, \mathcal{F}, m)$, $1 \leqq p \leqq \infty$, be the usual Banach spaces. For a set $A \in \mathcal{F}$, 1_A is the indicator function of A and $L_p(A)$ denotes the Banach space of all $L_p(X)$-functions that vanish on $X-A$. Let $\Gamma=\{T\}$ be a semigroup of positive linear operators on $L_1(X)$. Γ is called *bounded* if $\sup\{\|T\|_1 : T \in \Gamma\} < \infty$. Let $B(\Gamma)$ denote the space of all bounded real-valued functions on Γ. A *mean* φ on $B(\Gamma)$ is a linear functional on $B(\Gamma)$ such that

$$\inf\{b(T) : T \in \Gamma\} \leqq \varphi(b) \leqq \sup\{b(T) : T \in \Gamma\}$$

for all $b \in B(\Gamma)$. A mean φ on $B(\Gamma)$ is *left* [*right*] *invariant* if

$$\varphi({}_T b)=\varphi(b) \qquad [\varphi(b_T)=\varphi(b)]$$

for all $b \in B(\Gamma)$ and $T \in \Gamma$, where ${}_T b$ and b_T are the functions on Γ defined by ${}_T b(S)=b(TS)$ and $b_T(S)=b(ST)$ for all $S \in \Gamma$, respectively. An *invariant mean* is a left and right invariant mean. If $B(\Gamma)$ has a left [right] invariant mean, Γ is called *left* [*right*] *amenable.* If $B(\Gamma)$ has an invariant mean, then Γ is called *amenable.* It is well-known that commutative semigroups, solvable groups, locally finite groups, etc., are amenable (for these and more see Day [1]).

Recently the author [4] has proved that if $\Gamma=\{T\}$ is a bounded *left amenable* semigroup of positive linear operators on $L_1(X)$, then the following two conditions are equivalent: (0) There exists a strictly positive function $f_0 \in L_1(X)$ with $Tf_0=f_0$ for all $T \in \Gamma$; (i) $A \in \mathcal{F}$ and $m(A)>0$ imply $\inf\left\{\int_A T1\,dm : T \in \Gamma\right\}>0$. In the present paper we shall assume that Γ is a bounded *amenable* semigroup of positive linear operators on $L_1(X)$. Let us denote by IM the set of all invariant means on $B(\Gamma)$ and define, for $b \in B(\Gamma)$,

$$M(b)=\sup\{\varphi(b) : \varphi \in IM\}.$$

Then we have the following

Theorem. *Let $\Gamma=\{T\}$ be a bounded amenable semigroup of positive linear operators on $L_1(X)$. Then the following two conditions are* *equivalent*:

(0) *There exists a strictly positive function* $f_0 \in L_1(X)$ *with* $Tf_0 = f_0$ *for all* $T \in \Gamma$;

(ii) $A \in \mathcal{F}$ *and* $m(A) > 0$ *imply* $M\left(\int_A T1\, dm\right) > 0$.

For the proof of the Theorem we need the following decomposition similar to Sucheston's [5].

Lemma. Γ *decomposes the space* X *into two sets* Y *and* Z *such that*

(i) *if* $f \in L_1(Z)$ *then* $Tf \in L_1(Z)$ *for all* $T \in \Gamma$ *and* $\inf\{\|Tf\|_1 : T \in \Gamma\} = 0$,

(ii) *there exists a nonnegative function* $e \in L_\infty(Y)$ *with* $e > 0$ *on* Y *and* $T^*e = e$ *for all* $T \in \Gamma$, *where* T^* *denotes the adjoint of* T.

Proof. It is easy to see that there exists a nonnegative function $e \in L_\infty(X)$, with $T^*e = e$ for all $T \in \Gamma$, such that $0 \leqq u \in L_\infty(X)$ and $T^*u = u$ for all $T \in \Gamma$ imply $\operatorname{supp} u \subset \operatorname{supp} e$. Let $Y = \operatorname{supp} e$ and $Z = X - Y$, and let $0 \leqq f \in L_1(Z)$. Then, since $\langle Tf, e\rangle = \langle f, T^*e\rangle = \langle f, e\rangle = 0$, $Tf \in L_1(Z)$ for all $T \in \Gamma$. In order to prove that $\inf\{\|Tf\|_1 : T \in \Gamma\} = 0$, let $\varphi \in IM$ and define a positive linear functional Ψ on $L_1(X)$ by the relation:

$$\Psi(g) = \varphi\left(\int Tg\, dm\right) \qquad (g \in L_1(X)).$$

Since the dual space of $L_1(X)$ is the space $L_\infty(X)$, there exists a nonnegative function $u \in L_\infty(X)$ such that

$$\Psi(g) = \int ug\, dm$$

for all $g \in L_1(X)$. We now show that $T^*u = u$ for all $T \in \Gamma$. To see this, fix $S \in \Gamma$ arbitrarily. Then for any $g \in L_1(X)$ we have $\langle g, S^*u\rangle = \langle Sg, u\rangle = \Psi(Sg) = \varphi\left(\int T(Sg)\, dm\right) = \varphi\left(\int Tg\, dm\right) = \langle g, u\rangle$, where the fourth equality follows from the fact that $\varphi \in IM$. Hence $S^*u = u$. Therefore $\operatorname{supp} u \subset \operatorname{supp} e = Y$ and $\varphi\left(\int Tf\, dm\right) = \int fu\, dm = 0$ because $\operatorname{supp} f \subset Z$. This completes the proof of the Lemma.

Proof of the Theorem. (0) implies (ii): Obvious from Corollary 1 of the author [4].

(ii) implies (0): For $T \in \Gamma$ and $f \in L_1(Y)$, define $T'f = (Tf)1_Y$. By the Lemma, $T'S' = (TS)'$ for all $T, S \in \Gamma$ and $T'^*e = e$ for all $T' \in \Gamma'$. Let A be a measurable subset of Y with $m(A) > 0$. Since $T'^*1_A = T^*1_A$, we have

$$M\left(\int_A T'(1_Y)\, dm\right) = M\left(\int_A T1\, dm\right) > 0.$$

Hence it follows from Proposition 1 of [4] that

$$(1) \qquad \inf\left\{\int_A T1\,dm : T \in \Gamma\right\} = \inf\left\{\int_A T'(1_Y)dm : T' \in \Gamma'\right\} > 0$$

for every $A \in \mathcal{F}$ with $A \subset Y$ and $m(A) > 0$.

Let $\varphi \in IM$ and define a positive linear functional λ on $L_\infty(X)$ by the relation:

$$\lambda(u) = \varphi\left(\int u(T1)dm\right) \qquad (u \in L_\infty(X)).$$

If T^{**} denotes the adjoint of T^*, then for any $u \in L_\infty(X)$ and $S \in \Gamma$ we have

$$\begin{aligned} S^{**}\lambda(u) = \lambda(S^*u) &= \varphi\left(\int (S^*u)T1\ dm\right) \\ &= \varphi\left(\int u(ST1)dm\right) = \varphi\left(\int u(T1)dm\right) = \lambda(u). \end{aligned}$$

It follows that $S^{**}\lambda = \lambda$. Hence if μ denotes the maximal (countably additive) measure satisfying $0 \leqq \mu \leqq \lambda$ (cf. Neveu [3], Lemma 1), then $S^{**}\mu \leqq \mu$. Let $h = d\mu/dm$. It follows that $Sh \leqq h$. But, since $(\mu - S^{**}\mu)(e) = \mu(e) - \mu(S^*e) = 0$ and $e > 0$ on Y, we have $Sh = h$ on Y. Moreover it follows from (1) that $h > 0$ on Y. Therefore if μ_0 denotes the positive linear functional on $L_\infty(X)$ defined by the relation:

$$\mu_0(u) = \varphi\left(\int u(Th)dm\right) \qquad (u \in L_\infty(X)),$$

then μ_0 is a countably additive measure, and if we let $f_0 = d\mu_0/dm$ then $Tf_0 = f_0$ for all $T \in \Gamma$ and $f_0 > 0$ on Y. Let $F = X - \operatorname{supp} f_0$. To complete the proof of the Theorem it suffices to show that $m(F) = 0$. To do this, we note that if $f \in L_1(X-F)$ then $Tf \in L_1(X-F)$ for all $T \in \Gamma$. This is an easy consequence of the fact that $Tf_0 = f_0$ for all $T \in \Gamma$. It now follows that $T1_F = T1$ on F for all $T \in \Gamma$. Since $F \subset Z$, the Lemma implies that

$$\inf\{\|T1_F\|_1 : T \in \Gamma\} = 0.$$

Hence for any $\varphi' \in IM$ we have

$$0 \leqq \varphi'\left(\int_F T1\ dm\right) \leqq \varphi'\left(\int T1_F\ dm\right) \leqq 0,$$

and $m(F) = 0$ by condition (ii). The proof is complete.

In conclusion we note that the identification of M defined on $B(\Gamma)$ is studied by Granirer [2] in some detail.

References

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