58. A Family of Pseudo-Differential Operators and a Stability Theorem for the Friedrichs Scheme

By Zen'ichirô KOSHIBA*) and Hitoshi KUMANO-GO**)

(Comm. by Kôsaku YOSIDA, M. J. A., May 12, 1976)

§0. Introduction. In this note we shall study an algebra of a family of pseudo-differential operators and try to apply this theory to the stability theory of the Friedrichs scheme. The class $\{S^m_{\lambda_h}\}$ of pseudo-differential operators is defined by a family of basic weight functions $\lambda_h(\xi)$ $(0<h<1)$ as in [4], [5] and [2].

For the application to the stability theory we have to define two subclasses $\{\mathring{S}^m_{\lambda_h}\}$ and $\{\tilde{S}^m_{\lambda_h}\}$ of $\{S^m_{\lambda_h}\}$ as the sets of all the symbols $p_h(x,\xi)$ such that $h^{-1}p_h\in\{S^{m+1}_{\lambda_h}\}$ and $h^{-1}\partial^\alpha_\xi p_h\in\{S^{m+1-|\alpha|}_{\lambda_h}\}$ for any $\alpha\neq 0$, respectively. We have also to derive 'the principle of cutting off' a symbol $p_h(x,\xi)$ of class $\{S^m_{\lambda_h}\}$ by $\chi(\lambda_h(\xi))$ (or $\varphi(\zeta_h(\xi))$) (see Theorem 1.9). Then, we can treat difference schemes as a family of pseudo-differential operators, and prove a stability theorem of the Friedrichs schemes for a diagonalizable hyperbolic system. We note that this theorem is regarded as the general form of the Yamaguti-Nogi-Vaillancourt stability theorem in [7], [8] and [9], and note that the theorem holds without the restriction on the behavior of symbols $p_h(x,\xi)$ at $x=\infty$.

§1. A family of pseudo-differential operators.

Definition 1.1. A family $\{\lambda_h(\xi)\}_{0<h<1}$ of real valued C^∞-functions in R^n is called a basic weight function, when there exist positive constants A_0, A_α (independent of $0<h<1$) such that

(1.1) $1\leqq\lambda_h(\xi)\leqq A_0\langle\xi\rangle,\ |\lambda^{(\alpha)}_h(\xi)|\leqq A_\alpha\lambda_h(\xi)^{1-|\alpha|}$ for any α,

where $\langle\xi\rangle=\{1+|\xi|^2\}^{1/2}$, $\lambda^{(\alpha)}_h=\partial^\alpha_\xi\lambda_h$ for $\alpha=(\alpha_1,\cdots,\alpha_n)$.

Example. An important example of this note is defined by

(1.2) $\lambda_h(\xi)=\langle\zeta_h(\xi)\rangle,\ \zeta_h(\xi)=(h^{-1}\sin h\xi_1,\cdots,h^{-1}\sin h\xi_n)$ (see [4], [5]).

Definition 1.2. i) A family $\{p_h\}$ of C^∞-symbols $p_h(x,\xi)$ in $R^n_x\times R^n_\xi$ $(0<h<1)$ is called of class $\{S^m_{\lambda_h}\}$ $(-\infty<m<\infty)$, when there exist constants $C_{\alpha,\beta}$ (independent of $0<h<1$) such that

(1.3) $|p^{(\alpha)}_{h(\beta)}(x,\xi)|\leqq C_{\alpha,\beta}\lambda_h(\xi)^{m-|\alpha|}$ *for any* α,β,

where $p^{(\alpha)}_{h(\beta)}=\partial^\alpha_\xi D^\beta_x p_h$ $(D_x=-i\partial_x)$. We set $\{S^{-\infty}_{\lambda_h}\}=\bigcap_m\{S^m_{\lambda_h}\}$ and $\{S^{\infty}_{\lambda_h}\}=\bigcup_m\{S^m_{\lambda_h}\}$.

ii) A family $\{P_h\}$ of linear operators $P_h:\mathcal{S}\to\mathcal{S}$ is called a pseudo-differential operator of class $\{S^m_{\lambda_h}\}$ with symbol $p_h(x,\xi)$, when there

*) Department of Mathematics, Shinshû University.
**) Department of Mathematics, Osaka University.

exists a symbol $p_h(x,\xi)$ of class $\{S^m_{\lambda_h}\}$ such that

$$P_h u(x)=\int e^{ix\cdot\xi}p_h(x,\xi)\hat{u}(\xi)đ\xi \qquad \text{for } u\in\mathcal{S}, \tag{1.4}$$

where $đ\xi=(2\pi)^{-n}d\xi$, $\mathcal{S}$ denotes the Schwartz space, and $\hat{u}(\xi)=\int e^{-ix\cdot\xi}u(x)dx$. We denote it by $P_h=p_h(X,D_x)\in\{S^m_{\lambda_h}\}$, or $\sigma(P_h)(x,\xi)=p_h(x,\xi)$.

Example. $\lambda_h(\xi)^s\in\{S^s_{\lambda_h}\}$ for any real s, $\cos h\xi_j$, $\sin h\xi_j\in\{S^0_{\lambda_h}\}$. For $p(x,\xi)\in S^m_{\langle\xi\rangle}$, $p_h(x,\xi)=p(x,\zeta_h(\xi))\in\{S^m_{\lambda_h}\}$ (see [5]).

We have the following series of theorems (see [5]):

Theorem 1.3. i) *Let $P_h=p_h(X,D_x)\in\{S^m_{\lambda_h}\}$ and let define the formal adjoint P_h^* by*

$$(P_h u,v)=(u,P_h^* v) \qquad \text{for } u,v\in\mathcal{S}. \tag{1.5}$$

Then, $P_h^*\in\{S^m_{\lambda_h}\}$ and $\sigma(P_h^*)=p_h^*(x,\xi)$ has the asymptotic expansion

$$p_h^*(x,\xi)\sim\sum_\alpha\frac{(-1)^{|\alpha|}}{\alpha!}\overline{p_{h(\alpha)}^{(\alpha)}(x,\xi)} \tag{1.6}$$

in the sense $p_h^*-\sum_{|\alpha|<N}\frac{(-1)^{|\alpha|}}{\alpha!}\overline{p_{(\alpha)}^{(\alpha)}}\in\{S^{m-N}_{\lambda_h}\}$ for any N.

ii) *Let $P_{j,h}=p_{j,h}(X,D_x)\in\{S^{m_j}_{\lambda_h}\}$ $(j=1,2)$, and set $P_h=P_{1,h}P_{2,h}$. Then, $P_h\in\{S^{m_1+m_2}_{\lambda_h}\}$ and $\sigma(P_h)=p_h(x,\xi)$ has the asymptotic expansion*

$$p_h(x,\xi)\sim\sum_\alpha\frac{1}{\alpha!}p_{1,h}^{(\alpha)}(x,\xi)p_{2,h(\alpha)}(x,\xi). \tag{1.7}$$

By means of the relation (1.5) for $u\in\mathcal{S}'$ and $v\in\mathcal{S}$, we can extend $P_h:\mathcal{S}\to\mathcal{S}$ to the mapping $P_h:\mathcal{S}'\to\mathcal{S}'$ by Theorem 1.3-i).

Definition 1.4. We define the Sobolev space $H_{\lambda_h,s}$ by $H_{\lambda_h,s}=\{u\in\mathcal{S}'\,;\ \lambda_h(\xi)^s\hat{u}(\xi)\in L^2(R^n_\xi)\}$ with s-norm $\|u\|_{\lambda_h,s}=\|\lambda_h^s\hat{u}\|_{L^2}$.

Theorem 1.5. *For $P_h\in\{S^m_{\lambda_h}\}$ we have for constants C_s and l_s*

$$\|P_h u\|_{\lambda_h,s}\leqq(C_s|p_h|_{l_s}^{(m)})\|u\|_{\lambda_h,s+m} \qquad \text{for } u\in H_{\lambda_h,s+m}\,. \tag{1.8}$$

Here, $|p_h|_l^{(m)}$ are semi-norms defined by

$$|p_h|_l^{(m)}=\underset{|\alpha+\beta|\leqq l}{\mathrm{Max}}\ \sup_{(x,\xi)}\{|p_{h(\beta)}^{(\alpha)}(x,\xi)|/\lambda_h(\xi)^{m-|\alpha|}\}, \qquad l=0,1,\cdots. \tag{1.9}$$

Let $q(\sigma)$ $(\in C_0^\infty(R^n))$ be an even function such that $q(\sigma)\geqq 0$ and $\int q(\sigma)^2d\sigma=1$. For $p_h(x,\xi)\in\{S^m_{\lambda_h}\}$ we define a double symbol $q(\xi,x',\xi')$ by $q(\xi,x',\xi')=\int F(\xi,\zeta)p(x',\zeta)F(\xi',\zeta)d\zeta$, where $F(\xi,\zeta)=q((\zeta-\xi)/\lambda_h(\xi)^{1/2})\times\lambda_h(\xi)^{-n/4}$, and define the Friedrichs part $P_{F,h}$ of $P_h=p_h(x,D_x)$ by

$$\widehat{P_{F,h}u}(\xi)=\int e^{-ix'\cdot\xi}\left\{\int e^{ix'\cdot\xi'}q(\xi,x',\xi')\hat{u}(\xi')đ\xi'\right\}dx' \qquad \text{for } u\in\mathcal{S}. \tag{1.10}$$

Theorem 1.6. *Let $P_h=p_h(X,D_x)\in\{S^m_{\lambda_h}\}$. Then, we have*

$$P_{F,h}\in\{S^m_{\lambda_h}\}, \qquad p_{F,h}(x,\xi)-p_h(x,\xi)\in\{S^{m-1}_{\lambda_h}\}. \tag{1.11}$$

If $P_h=p_h(X,D_x)$ $(\in\{S^m_{\lambda_h}\})$ is an $l\times l$ matrix and $p_h(x,\xi)$ is hermitian symmetric, then we have

$$(P_{F,h}u,v)=(u,P_{F,h}v) \qquad \text{for } u,v\in\mathcal{S}. \tag{1.12}$$

Furthermore, if $p_h(x,\xi)\geqq c_o\lambda_h(\xi)^m I$ *for a real* c_o, *then we have*

(1.13) $\quad (P_{F,h}u,u)\geqq c_o\|u\|^2_{\lambda_h,m/2}$ *for* $u\in\mathcal{S}$ (*see* [6], [3], [5]).

Definition 1.7. i) We say that $p_h(x,\xi)$ ($\in\{S^m_{\lambda_h}\}$) belongs to a class $\{\mathring{S}^m_{\lambda_h}\}$, when $h^{-1}p_h\in\{S^{m+1}_{\lambda_h}\}$, and say that $p_h(x,\xi)$ ($\in\{S^m_{\lambda_h}\}$) belongs to a class $\{\tilde{S}^m_{\lambda_h}\}$, when $h^{-1}p_h^{(\alpha)}\in\{S^{m+1-|\alpha|}_{\lambda_h}\}$ for any $\alpha\neq 0$.

Example. $\sin h\xi_j\in\{\mathring{S}^0_{\lambda_h}\}$, $\cos h\xi_j\in\{\tilde{S}^0_{\lambda_h}\}$, $p(x,h\xi)\in\tilde{S}^0_{\lambda_h}$ for $p\in\mathcal{B}^\infty(R^{2n}_{x,\xi})$ and $\{\mathring{S}^m_{\lambda_h}\}\subset\{\tilde{S}^m_{\lambda_h}\}$.

Theorem 1.8. *For* $P_{j,h}\in\{\tilde{S}^{m_j}_{\lambda_h}\}$ $(j=1,2)$ *we have* $[P_{1,h},P_{2,h}]\in\{\mathring{S}^{m_1+m_2-1}_{\lambda_h}\}$, *and for* $P_{1,h}=p_{1,h}(D_x)\in\{\tilde{S}^{m_1}_{\lambda_h}\}$ *and* $P_{2,h}\in\{S^{m_2}_{\lambda_h}\}$ *have* $[P_{1,h},P_{2,h}]\in\{\mathring{S}^{m_1+m_2-1}_{\lambda_h}\}$.

Theorem 1.9 (the principle of cutting off). *Let* $\chi(t)$ *and* $\varphi(\xi)$ *be* C^∞_0*-functions in* R^1 *and* R^n, *respectively. Then, we have* $\chi_h(\xi)=\chi(\lambda_h(\xi))$, $\varphi_h(\xi)=\varphi(\zeta_h(\xi))\in\{S^{-\infty}_{\lambda_h}\}$. *If* $p_h(x,\xi)\in\{S^m_{\lambda_h}\}$ (*or* $\in\{\mathring{S}^m_{\lambda_h}\}$), *then we have* χ_hp_h, $\varphi_hp_h\in\{S^{-\infty}_{\lambda_h}\}$ (*or* χ_hp_h, $\varphi_hp_h\in\mathring{S}^{-\infty}_{\lambda_h}$).

Theorem 1.10. *Let* $P_h=p_h(X,D_x)\in\{S^{m_1}_{\lambda_h}\}$ *be an* $l\times l$ *matrix such that* $p_h(x,\xi)\geqq 0$, *and let* $q_h(\xi)\in\{\tilde{S}^{m_2}_{\lambda_h}\}$ *be a scalar symbol. Then we have for a constant* C

(1.14) $\quad (P_{F,h}q(D_x)^2u,u)\geqq -Ch\|u\|^2_{\lambda_h,m_1/2+m_2}$ *for* $u\in\mathcal{S}$.

§2. A stability theorem for the Friedrichs scheme. Consider the hyperbolic system of the form

$$(2.1)\qquad \begin{cases} Lu=D_tu-p(t,X,D_x)u=0 & \text{in } [0,T]\times R^n\ (T>0),\\ u|_{t=0}=u_o\in L^2(R^n) & \text{for } u=(u_1,\cdots,u_l),\end{cases}$$

where $p(t,X,D_x)\in\mathcal{B}_t(S^1_{\langle\xi\rangle})$ on $[0,T]$ (i.e., $p(t,x,\xi)$ is a $S^1_{\langle\xi\rangle}$-valued C^∞-function of t on $[0,T]$). We assume that $p(t,x,\xi)$ has the form

(2.2) $\quad p(t,x,\xi)=p_1(t,x,\xi)+p_0(t,x,\xi)$ $(p_j\in\mathcal{B}_t(S^j_{\langle\xi\rangle}),\ j=0,1)$,

and that all the eigenvalues $\mu_j(t,x,\xi)$ $(j=1,\cdots,l)$ of p_1 are real and for constants μ_o and $M_o>0$ we have

(2.3) $\quad \underset{(t,x,\xi)}{\mathrm{Max}}|\mu_j(t,x,\xi)|\leqq\mu_o|\xi|$ on $[0,T]\times R^n_x\times\{|\xi|\geqq M_o\}$ $(j=1,\cdots,l)$.

We also assume that $p_1(t,x,\xi)$ is diagonalizable in the sense: there exists $N(t,x,\xi)\in\mathcal{B}_t(S^0_{\langle\xi\rangle})$ such that

(2.4) $\quad N(t,x,\xi)p_1(t,x,\xi)=\mathcal{D}(t,x,\xi)N(t,x,\xi)$ on $[0,T]\times R^n_x\times\{|\xi|\geqq M\}$

and

(2.5) $\quad |\det(N(t,x,\xi))|\geqq c_o$ on $[0,T]\times R^n_x\times\{|\xi|\geqq M\}$

for constants $M(\geqq M_0)$, $c_o>0$ and $\mathcal{D}=\begin{pmatrix}\mu_1(t,x,\xi) & & 0\\ & \ddots & \\ 0 & & \mu_l(t,x,\xi)\end{pmatrix}$.

For the operator L we define the Friedrichs schemes $S_h=S_h(t)$ by

(2.6) $\quad \sigma(S_h)(t,x,\xi)=q_h(\xi)-i\tau hp_h(t,x,\xi)$ for a real fixed τ

$$\left(q_h(\xi)=n^{-1}\sum_{j=1}^n\cos h\xi_j,\ p_h(t,x,\xi)=p(t,x,\zeta_h(\xi))\right).$$

Then, applying Theorem 1.6, Theorems 1.8–1.10, we have

Theorem 2.1. *Assume* $|\tau|\leqq(\sqrt{n}\,\mu_0)^{-1}$. *Then, there exist constants* C, $C'>0$ *such that we have for* $(S_h)^\nu=\prod_{j=0}^{\nu-1} S_h(j\tau h)$

$$(2.7)\qquad \|S_h u\|^2_{H_h}-\|u\|^2_{H_h}\leqq Ch\,\|u\|^2_{L^2},\qquad \|(S_h)^\nu u\|_{L^2}\leqq C'\,\|u\|_{L^2}\ (0\leqq\tau h\nu\leqq T).$$

Here, $\|u\|^2_{H_h}=(H_h u,u)$ *and* H_h *is the Friedrichs part of* $\tilde{H}_h$ *defined by*

$$\sigma(\tilde{H}_h)(t,x,\xi)=N^*(t,x,\zeta_h(\xi))N(t,x,\zeta_h(\xi))(1-\varphi_h(\xi))^2+\varphi_h(\xi)I.$$

The detailed proof will be published elsewhere.

References

[1] A. P. Calderón and R. Vaillancourt: A class of bounded pseudo-differential operators. Proc. Nat. Acad. Sci. USA, **69**, 1185–1187 (1972).

[2] L. S. Frank: Algèbre des opérateurs aux différences finies. Israel J. Math., **13**, 24–55 (1972).

[3] K. O. Friedrichs: Pseudo-Differential Operators. Lecture Notes, Courant Inst. Math. Sci., New York Univ. (1968).

[4] H. Kumano-go: Pseudo-differential operators and the uniqueness of the Cauchy problem. Comm. Pure Appl. Math., **22**, 73–129 (1969).

[5] H. Kumano-go: Pseudo-Differential Operators. Iwanami Shoten, Tokyo (1974) (in Japanese).

[6] P. D. Lax and L. Nirenberg: On stability for difference schemes: a sharp form of Gårding's inequality. Comm. Pure Appl. Math., **19**, 473–492 (1966).

[7] R. Vaillancourt: A strong form of Yamaguti and Nogi's stability theorem for Friedrichs' scheme. Publ. Res. Inst. Math. Sci., **5**, 113–117 (1969).

[8] M. Yamaguti: On the pseudo-difference schemes. Proc. Internat. Conf. on Functional Analysis and Related Topics, 161–170 (Tokyo, 1969), Univ. of Tokyo Press, Tokyo (1970).

[9] M. Yamaguti and T. Nogi: An algebra of pseudo difference schemes and its application. Publ. Res. Inst. Math. Sci., Ser., A **3**, 151–166 (1967).