76. *On Some Additive Divisor Problems. II*

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§1. In our previous paper [4] we have given a very simple proof of the asymptotic formula (as $N\to\infty$)

$$\mathrm{D}_k(N;a)=\sum_{n\leqq N}\mathrm{d}_2(n+a)\mathrm{d}_k(n)$$
$$=S_k(a)N(\log N)^k+O(N(\log N)^{k-1}\operatorname{loglog}N),$$

where a is a fixed integer, $\mathrm{d}_k(n)$ the coefficient of $\zeta(s)^k$, $k\geqq 3$ arbitrary. The problem for general k has been firstly treated by Linnik in his book [3]. There it is indicated also that his method enables us to deduce even an expansion with an error-term $O(N(\log N)^\varepsilon)$, $\varepsilon>0$ being arbitrarily small (see also Bredikhin [1]). But it seems that neither Linnik nor Bredikhin have been able to eliminate this error-term.

Now the purpose of this note is to announce

Theorem. *We have the asymptotic expansion, for arbitrary k,*

$$\mathrm{D}_k(N;a)=N\sum_{j=0}^{k}f_k^{(j)}(a)(\log N)^j+O(N(\log N)^{-1+\varepsilon}).$$

The coefficients can be calculated, but at the cost of big labour. The result should be compared with Estermann's asymptotic expansion for the case of $k=2$ ([2]).

§2. We indicate very briefly the main steps of our proof, whose detailed exposition will appear elsewhere.

Now by an obvious reason it is sufficient to consider the case of $a=1$. And we prove that, denoting by (P) the set of primes in the interval $(N^{3/4}, N(\log N)^{-A})$ with sufficiently large A, we have

$$(*)\qquad \mathrm{D}_k(N;1)-\mathrm{D}_k(N;p)=O(N(\log N)^{-1+\varepsilon}),$$

uniformly for all $p\in(P)$. To do this we divide $\mathrm{D}_k(N;a)$ into two parts. Let $z_1=\exp((\log N)^{\varepsilon_1})$, $\varepsilon_1=\varepsilon/(3k+1)$, $z_2=\exp((\log N)(\operatorname{loglog}N)^{-2})$, and further let (I), (II) be two sets of integers $\leqq N$ such that $n\in$ (I) has no prime factors in the interval (z_1,z_2) and (II) is the complementary set of (I). And we put, a being 1 or $p\in(P)$,

$$\mathrm{D}_k(N;a)=\sum_{n\in(\mathrm{I})}+\sum_{n\in(\mathrm{II})}=\mathrm{D}_k^{(1)}(N;a)+\mathrm{D}_k^{(2)}(N;a).$$

By a direct application of the dispersion method [3] we can show that

Lemma 1. *We have, uniformly for all $p\in(P)$,*

$$\mathrm{D}_k^{(2)}(N;1)-\mathrm{D}_k^{(2)}(N;p)=O(N(\log N)^{-2}).$$

§3. As for $\mathrm{D}_k^{(1)}(N;a)$ we first define another two sets of integers

Δ_1 and Δ_2. Δ_1 consists of all integers whose prime factors are all less than z_1, and Δ_2 consists of all integers whose prime factors are all larger than z_2. Then we have, after an elementary consideration

$$\mathrm{D}_k^{(1)}(N;a)=\sum_{\substack{r\leqq z_1'\\ r\in\Delta_1}}\mathrm{d}_k(r)\sum_{\substack{n\leqq N/r\\ n\in\Delta_2}}\mathrm{d}_2(rn+a)\mathrm{d}_k(n)+O(N\exp(-(\log N)^{\varepsilon_1})),$$

where $z_1'=\exp((\log N)^{3\varepsilon_1})$. Let then $\mathrm{T}_k(N;r,a)$ be the inner sum. We can write this in the following form:

$$\mathrm{T}_k(N;r,a)=2\sum_{\substack{q\leqq Q_1(r)\\ (q,r)=1}}\sum_{\substack{n\equiv-a\bar{r}\ (\mathrm{mod}\ q)\\ n\leqq N/r\\ n\in\Delta_2}}\mathrm{d}_k(n)+O\Bigg(\sum_{\substack{Q_1(r)<q<Q_2(r)\\ (q,r)=1}}\sum_{\substack{n\equiv-a\bar{r}\ (\mathrm{mod}\ q)\\ n\leqq N/r\\ n\in\Delta_2}}\mathrm{d}_k(n)\Bigg),$$

where $r\bar{r}\equiv1\ (\mathrm{mod}\ q)$, $Q_1(r)=(N/r)^{1/2}(\log(N/r))^{-E}$, $Q_2(r)=(Nr)^{1/2}(\log(N/r))^{E}$ with sufficiently large E. And here we have

Lemma 2. *Let*

$$\mathrm{U}_k(x;q,t)=\sum_{\substack{n\equiv t\ (\mathrm{mod}\ q)\\ n\leqq x\\ n\in\Delta_2}}\mathrm{d}_k(n),\qquad (q,t)=1.$$

Then we have

$$\mathrm{U}_k(x;q,t)=O(x(\log\log N)^{c(k)}/(q\log N)),$$

uniformly for any $q\leqq x^{1-\varepsilon}$, $N^{3/4}\leqq x\leqq N$, *where* $c(k)$ *depends on* k *and* ε *at most.*

Lemma 3. *Let*

$$\mathrm{E}_k(x;q)=\max_{y\leqq x}\max_{(q,t)=1}\Bigg|\mathrm{U}_k(y;q,t)-\sum_{\substack{n\leqq y\\ (n,q)=1\\ n\in\Delta_2}}\mathrm{d}_k(n)/\varphi(q)\Bigg|.$$

Then we have, if $N^{3/4}\leqq x\leqq N$,

$$\sum_{q\leqq x^{1/2}(\log x)^{-B_1}}\mathrm{E}_k(x;q)=O(x(\log N)^{-B}),$$

where B, B_1 *can be arbitrarily large but* B_1 *depends on* B.

Lemma 2 can be proved by an idea of Wolke [5], and Lemma 3 can be deduced by an application of the large sieve method as in our previous paper [4].

Hence we see at once that, uniformly for any $p\in(P)$,

$$\mathrm{T}_k(N;r,1)-\mathrm{T}_k(N;r,p)=O(N(\log\log N)^{c(k)}(\log r)/(r\log N)),$$

and so

$$\mathrm{D}_k(N;1)-\mathrm{D}_k(N;p)=O(N(\log\log N)^{c(k)}(\log N)^{-1}\sum_{r\leqq z_1'}\mathrm{d}_k(r)(\log r)/r)$$
$$=O(N(\log N)^{-1+(3k+1)\varepsilon_1}).$$

Thus from Lemma 1 we get (*). Then, denoting by $\pi((P))$ the number of elements in (P), we have

$$\mathrm{D}_k(N;1)=\pi((P))^{-1}\sum_{p\in(P)}\mathrm{D}_k(N;p)+O(N(\log N)^{-1+\varepsilon}).$$

And the last sum over p can be calculated by the trigonometrical method of I. M. Vinogradov.

References

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