74. *An Induction Principle for the Generalization of Bombieri's Prime Number Theorem*

By Yoichi MOTOHASHI
Department of Mathematics, College of Science and Technology, Nihon University, Tokyo

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1. Recently several authors have considered generalizations into various directions of Bombieri's prime number theorem [2]. Here we give an induction principle through which most of former results follow in improved forms and also with which we can expand considerably the domain of the equi-distributed sequences (for this terminology see [1]).

Let f be a complex valued arithmetic function, and let introduce the following properties. $(\mathcal{A})$: $f(n)=O(\tau(n)^C)$, where $\tau(n)$ is the divisor function. $(\mathcal{B})$: If the conductor of a non-principal character χ is $O((\log x)^D)$, then we have $\sum_{n\leqq x} f(n)\chi(n)=O(x(\log x)^{-3D})$. Further we consider the equi-distribution property

$$(\mathcal{C}):\ \sum_{q\leqq x^{1/2}(\log x)^{-B}} \max_{y\leqq x} \max_{(q,l)=1} |E(y;q,l;f)|=O(x(\log x)^{-A}),$$

where

$$E(y;q,l;f)=\sum_{\substack{n\equiv l \,(\mathrm{mod}\, q)\\ n\leqq y}} f(n)-\varphi(q)^{-1}\sum_{\substack{(n,q)=1\\ n\leqq y}} f(n),$$

$\varphi(q)$ being the Euler function. In the above it is understood that C is a fixed constant and $A, B=B(A)$, D can be taken arbitrarily large and that these are all depending only on f. Then we have

Theorem 1. *Let f and g have the properties $(\mathcal{A})$, $(\mathcal{B})$, $(\mathcal{C})$. Then the multiplicative convolution $f*g$ does so.*

2. As for the proof of Theorem 1 we remark the following equality: If $y\leqq x$ and $(q,l)=1$, then

$$\begin{aligned} E(y;q,l;f*g)=&\sum_{\substack{(u,q)=1\\ (\log x)^K\leqq u\leqq x(\log x)^{-K'}}} f(u)E(y/u;q,l\bar{u};g)\\ &+\sum_{\substack{(u,q)=1\\ u<(\log x)^K}} f(u)E(y/u;q,l\bar{u};g)\\ &+\sum_{\substack{(v,q)=1\\ v\leqq(\log x)^{K'}}} g(v)\{E(y/v;q,l\bar{v};f)\\ &\qquad -E(\min(y/v,x(\log x)^{-K});q,l\bar{v};f)\}\\ =&\textstyle\sum_1+\sum_2+\sum_3, \qquad \text{say,}\end{aligned}$$

where $u\bar{u}\equiv1$, $v\bar{v}\equiv1 \pmod q$ and K, K' are to be taken appropriately.

The sums $\sum_2$ and $\sum_3$ are readily estimated by the property ($\mathcal{C}$). And the second Riesz mean of $\sum_1$ is expressed as

$$\varphi(q)^{-1} \sum_{\chi \neq \chi_0 \ (\mathrm{mod}\ q)} \frac{\bar{\chi}(l)}{2\pi i} \int_{(\alpha)} \left\{ \sum_{(\log x)^K \leqq n \leqq x (\log x)^{-K'}} \chi(n) f(n) n^{-s} \right\} \times \left\{ \sum_{n=1}^{\infty} \chi(n) g(n) n^{-s} \right\} \frac{y^s}{s^3} ds$$

where χ_0 is the principal character (mod q) and the integral is along the line $\mathrm{Re}(s) = \alpha > 1$. In the smoothening procedure we use the property ($\mathcal{A}$). For the chatacters with relatively small conductor (of the order of a power of $\log x$) we can use the property ($\mathcal{B}$). And for the characters with larger conductor we appeal to the large sieve method coupled with the device of Chen [3] on a dividing of integrand. However it should be stressed that the certain obstacle caused by imprimitive characters which are observed in Chen's argument does not appear at all in our procedure. So we get at the same time a simplified proof of Chen's theorem on the binary Goldbach problem ([3]). The detailed account will appear elsewhere.

3. Now we turn to some easy applications of Theorem 1. The function $f \equiv 1$ has obviously the three properties above, so by induction we get the analogue for $\tau_k(n)$ the k-th divisor function of Bombieri's theorem. Appealing to Bombieri's theorem we get a strengthening of Theorem 1 of [4]. In the same way Saz 1 of [5] will expand the scope of our theorem. Also it is easy to see that Theorem 1 allows us to iterate Bombieri's theorem, and we state it as a theorem because of its independent interest:

Theorem 2. *Let $a \geqq 1$ be a fixed integer. Let δ_j and η_j ($j = 1, 2, \cdots, a$) be non-negative numbers such that $\delta_j > \eta_j$ for each j. And $w(n) = w(n; \delta, \eta, a)$ is defined to be 1, if n can be written as $n = p_1 p_2 \cdots p_a$ where $n^{\delta_j} > p_j > n^{\eta_j}$ for each j, and to be zero, otherwise. Then the function $w(n)$ has the property ($\mathcal{C}$) uniformly for any choice of δ's and η's.*

This contains as special cases Satz of [6], estimations needed in Chen's argument [3] as well as the improved form of Theorem 1′ and 1″ of [4]. We also have applications of Theorem 2 to the additive divisor problems. Among other things we have the asymptotic expansions (as $N \to \infty$):

$$\sum_{p_1 p_2 \cdots p_a \leqq N} \tau(p_1 p_2 \cdots p_a - 1) = N \sum_{j=0}^{a-1} d_j^{(a)} (\log\log N)^j + O(N (\log\log N)^a / (\log N)),$$

$$\sum_{p_1 p_2 \cdots p_a \leqq N} r(p_1 p_2 \cdots p_a - 1) = N \sum_{j=0}^{a-1} e_j^{(a)} (\log\log N)^j / (\log N) + O(N (\log N)^{-1-\kappa}),$$

where $\kappa > 0$ is a certain constant and $r(n)$ is the number of representations of n as a sum of two squares. The calculations of $d_{a-1}^{(a)}$ and $e_{a-1}^{(a)}$

are not difficult while the other coefficients need much labour. Similar results (but only for $a=2$ and with inferior error-term) have been announced in [4].

References

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