73. *Group Rings of Metacyclic p-Groups*

By Shigeo KOSHITANI
Department of Mathematics, Tokyo University of Education

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Let K be a field with characteristic $p>0$, P a finite p-group and KP a group ring of P over K. Recently W. Müller [6] proved that every left ideal of KP is generated by at most 2 elements if $p=2$ and P is either a dihedral group, a semi-dihedral group or a generalized quaternion group of order 2^{n+1}. These groups are metacyclic 2-groups. So in this paper we shall generalize the above result as follows: If P is a metacyclic p-group containing a cyclic normal subgroup Q and with a cyclic factor group P/Q, then every left (right) ideal of KP is generated by at most $|P/Q|$ elements. Further we shall show that there exists a metacyclic p-group P such that KP has a left (right) ideal whose minimal generators consist of $|P/Q|$ elements. By using our technique if P is a semi-direct product of Q by P/Q it is proved a relation among the nilpotency indices of the radicals of KP, KQ and $K(P/Q)$ which is similar in the case of a direct product of groups.

Let P be a metacyclic p-group containing a cyclic normal subgroup $Q=[b]$ of order p^n $(n\geqq 1)$ and with a cyclic factor group $P/Q=[aQ]$ of order p^m (cf. [1, §47]). Then there is an integer r such that $aba^{-1}=b^r$. Since $a^{p^m}\in Q$, $r^{p^m}\equiv 1 \pmod{p^n}$. Hence

$$(*)\qquad ba^i=a^i b^{r^{p^m-i}},\qquad \text{for } i=0,\cdots,p^m-1.$$

We may put $a^{p^m}=b^{p^k}$, $(0\leqq k\leqq n)$. Put $B=KP$, $x=a-1$, $y=b-1$ in B and $[s,t]=x^{p^m-1-s}y^{p^n-1-t}$, for $s=0,\cdots,p^m-1$; $t=0,\cdots,p^n-1$. Then $C=\{[s,t]\,|\,0\leqq s\leqq p^m-1,\ 0\leqq t\leqq p^n-1\}$ forms a K-basis of B. Next we make C a totally ordered set by introducing in the following way: $[s,t]<[s',t']$ if and only if $t<t'$, or $t=t'$ and $s<s'$. Since each $u\in B\backslash\{0\}$ can be expressed uniquely in the form $u=\sum_{i=1}^{d}k_{iu}c_{iu}$, where $k_{iu}\in K\backslash\{0\}$, $c_{iu}\in C$, for $i=1,\cdots,d$ and $c_{1u}<c_{2u}<\cdots<c_{du}$, we can define a map $h: B\backslash\{0\}\to C$ such that $h(u)=c_{du}$. Put $\binom{i}{j}=0$ if $i<j$ or $j<0$.

At first we shall prove the following

Lemma. (a) $x[s,t]=[s-1,t]$, *for* $s=1,\cdots,p^m-1$; $t=0,\cdots,p^n-1$.

(b) $x[0,t]=0$, *for* $t=0,\cdots,p^k-1$. $x[0,t]=[p^m-1,t-p^k]$, *for* $t=p^k, p^k+1,\cdots,p^n-1$, if $k<n$.

(c) $y[p^m-1,t]=[p^m-1,t-1]$, *for* $t=1,\cdots,p^n-1$. $y[p^m-1,0]=0$.

(d) $h(y[s,t])=[s,t-1]$, *for* $s=0,\cdots,p^m-1$; $t=1,\cdots,p^n-1$. $y[s,0]=0$, *for* $s=0,\cdots,p^m-1$.

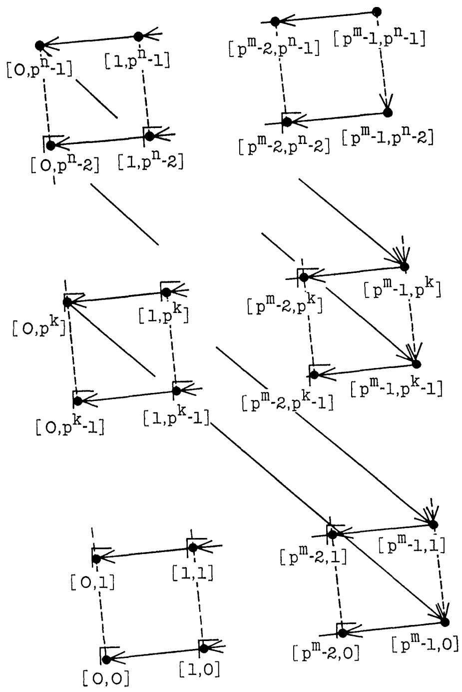
Proof. (a), (b) and (c) are clear. By (*), for $s=0,\cdots,p^m-1$,

$$yx^s=\sum_{i=0}^{s}\binom{s}{i}(-1)^{s-i}(ba^i-a^i)=\sum_{i=0}^{s}\binom{s}{i}(-1)^{s-i}(a^ib^{rp^m-i}-a^i)$$

$$=\sum_{j=0}^{s}\sum_{v=1}^{rp^m}\left\{\sum_{i=0}^{s}(-1)^{s-i}\binom{s}{i}\binom{i}{j}\binom{rp^m-i}{v}\right\}x^jy^v=x^sy+\sum_{j=0}^{s}\sum_{v=2}^{rp^m}a_{jv}x^jy^v,$$

where a_{jv} are elements of K.

We shall show the formulas in Lemma as diagrams,



where the diagram $u\to v$ means $xu=v$, the diagram $u \searrow v$ means $yu=v$ and the diagram $u \searrow v$ (dotted) means $h(yu)=v$.

Let us put $h(U)=\max\{h(u)\mid u\in U, u\neq 0\}$ for each left ideal $U\neq 0$ of B. Then we have the main theorem,

Theorem. *Let K be a field with characteristic $p>0$, and P a metacyclic p-group containing a cyclic normal subgroup Q and with a cyclic factor group P/Q. Then every left ideal of KP as well as every right ideal of KP is generated by at most $|P/Q|$ elements.*

In fact for a left ideal $U\neq 0$ of KP such that $h(U)=[s,t]$ U is generated by at most p^m-s elements.

Proof. We shall prove by induction on $s'=p^m-s$. If $s'=1$, $U=B[p^m-1,t]$ by Lemma. Assume that Theorem is proved for $1,\cdots,s'-1$. Since $h(U)=[s,t]$ there is $u_1\in U$ such that $h(u_1)=[s,t]$. Put

$$L=\sum_{i=0}^{s}\sum_{j=0}^{t} Kx^iy^ju_1+\sum_{v=s+1}^{p^m-1}\sum_{w=0}^{t-1} K[v,w].$$

By Lemma $[i,j]\in L$ for $i=0,\cdots,s$ and $j=0,\cdots,t$. Hence $U\subseteqq L$. So each $u\in U$ can be expressed uniquely in the form

$$u=\sum_{i=0}^{s}\sum_{j=0}^{t} c_{ij}^{(u)}x^iy^ju_1+\sum_{v=s+1}^{p^m-1}\sum_{w=0}^{t-1} d_{vw}^{(u)}[v,w],$$

where $c_{ij}^{(u)}$, $d_{vw}^{(u)}$ are elements of K. Put $D_v=\{d_{vw}^{(u)}\mid u\in U, w=0,\cdots,t-1\}$ for $v=s+1,s+2,\cdots,p^m-1$. If $D_v=0$ for all v, $U=Bu_1$. Assume that $D_v\neq 0$ for some v. Let W be a left ideal of B generated by a set

$$\{\sum_{v=s+1}^{p^m-1}\sum_{w=0}^{t-1} d_{vw}^{(u)}[v,w]\mid u\in U\}.$$

Hence $U=Bu_1+W$. Since $D_v\neq 0$ for some v, by Lemma, $h(W)=[s+1,*],[s+2,*],\cdots$, or $[p^m-1,*]$. From the hypothesis of induction W is generated by at most $s'-1$ elements. This proves Theorem.

Remark 1. There exists a metacyclic p-group P such that KP has a left ideal U whose minimal generators consist of p^m-s elements, where s is an integer such that $h(U)=[s,t]$.

In case $m\leqq k$ (hence $m\leqq n$) and $y[s',t']=[s',t'-1]$ for $s'=0,\cdots,p^m-1$; $t'=0,\cdots,p^m-1$, for each s $(0\leqq s\leqq p^m-1)$ a left ideal

$$U=B[s,p^m-1-s]+B[s+1,p^m-2-s]+\cdots+B[p^m-1,0]$$

is never generated by fewer than p^m-s-1 elements.

The above case happens if $m=1\leqq k\leqq n=z+1$ and $r=p^z+1$, where z is a positive integer, or if $m\leqq k$ and P is abelian.

Remark 2. Let P be a metacyclic p-group and denote by $J(KP)$ the radical of KP. Then $J(KP)=KPx+KPy=xKP+yKP$.

Remark 3 (cf. [3, Théorème 6], [5, Theorem]). Let P be a semi-direct product of $[b]$ of order p^n by $[a]$ of order p^m. Denote by $t(G)$ the nilpotency index of $J(KG)$ for any finite group G. Put $J(KP)^0=KP$ and $C_i=\{x^sy^t\mid 0\leqq s\leqq p^m-1, 0\leqq t\leqq p^n-1, s+t\geqq i\}$ for $i=0,\cdots,p^m+p^n-2$. Then we have,

(a) C_i forms a K-basis of $J(KP)^i$, for $i=0,\cdots,p^m+p^n-2$.

(b) $t(P)=t([a])+t([b])-1$.

By [4, Theorem 2, Theorem 7] and Theorem, we obtain,

Corollary (cf. [2, IV § 4]). *Let K be an algebraically closed field with characteristic $p>0$, G a finite p-nilpotent group with a metacyclic p-Sylow subgroup, $\{B_1,\cdots,B_m\}$ the set of all blocks of KG, and P_i a p-defect group of B_i such that P_i contains a cyclic normal subgroup Q_i and with a cyclic factor group P_i/Q_i for each i. Then every two-sided ideal of KG is generated by at most* $\max\{|P_i/Q_i|\mid i=1,\cdots,m\}$ *elements as a left ideal and as a right ideal.*

Remark 4. Let K be an algebraically closed field with characteristic $p>0$, G a finite group with a p-Sylow subgroup P and H the largest normal subgroup of G such that $p\nmid |H|$. If HP is normal in G and P is metacyclic, by [4, § 3] and Remark 2, $J(KG)$ is generated by at most 2 elements as a left ideal and as a right ideal.

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