96. *Basic Elements over Von Neumann Regular Rings*

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In [4], R. Wiegand and S. Wiegand have shown the following theorem: *Let R be a commutative regular ring. If Spec(R) has no 3-points, then every two-generator faithful R-module has a basic element. The converse holds if R is a Boolean ring.*

They have then asked that (1) for any commutative regular ring, whether the converse holds or not, and (2) whether "two-generator" can be replaced by "finitely generated" or not.

The purpose of this note is to answer (1) in the negative and (2) in the affirmative.

Throughout this note we assume that R is a commutative regular ring with identity 1 and R-modules are unital. We denote by $Q(R)$, $C(R)$ and $X(R)$ the maximal ring of quotients of R, the subring of $Q(R)$ generated by all idempotents in $Q(R)$ over R and the spectrum of R consisting of all prime ideals of R, respectively. For an element a in an R-module A and x in $X(R)$ we denote $a+Ax$ in A/Ax by a_x.

Let A be a finitely generated R-module. An element a in A is called basic in A if, for any x in $X(R)$, the image of a in A_x is part of a minimal generating set of A_x([3]). Since R is regular, as is well known, $R/x \simeq R_x$ and $A/Ax \simeq A_x$ for x in $X(R)$. So a in A is basic iff $a_x \neq 0$ for all $x \in X(R)$.

A point x in a topological space is called an n-point if there are pairwise disjoint open sets $U_1, \cdots, U_n$ such that $x \in U_i - U_i^-$ for $i=1, \cdots, n$, where U_i^- denotes the closure of U_i ([2]).

Lemma 1. *Let $e_1, \cdots, e_n$ be idempotents in R. Then there are orthogonal idempotents $f_1, \cdots, f_m$ in R such that $Re_1 + \cdots + Re_n = Rf_1 + \cdots + Rf_m$, and $e_i f_j = 0$ or f_j any i and j.*

Lemma 2 ([1]). *For x in $X(R)$,*

(i) *in case x is a non-isolated point, it is an n-point iff $[C(R)_x : R_x] \geq n$, the rank of $C(R)_x$ over R_x, and*

(ii) *if x is an isolated point, we have $[C(R)_x : R_x] = 1$.*

Lemma 3. *Let A be a finitely generated R-module. If, for any x in $X(R)$, there is a neighborhood N of x and an element a in A such that $a_z \neq 0$ for all z in N, then A has a basic element.*

Lemma 4. *If a finitely generated R-module A is faithful, then the factor module of A by its singular submodule is also faithful.*

The following theorem allows us to answer the question (2) in the affirmative.

Theorem 5. *If $X(R)$ contains no 3-points, then every finitely generated faithful R-module has a basic element.*

Proof. Let $A=Ra_1+\cdots+Ra_n$ be a finitely generated faithful R-module. We may assume that A is non-singular by Lemma 4. Hence, as is well known, it is embedded into a direct product $Q(R)\times\cdots\times Q(R)$ of n copies of $Q(R)$. Denote each a_i in $Q(R)\times\cdots\times(R)$ as $a_i=(p_{i1},\cdots,p_{in})$ with p_{ij} in $Q(R)$. Then, since A is faithful, it is easy to verify that $Q(R)=\sum_{i,j} Q(R)p_{ij}$. Take p'_{ij} in $Q(R)$ such that $p_{ij}p'_{ij}p_{ij}=p_{ij}$, and put $f_{ij}=p_{ij}p'_{ij}$ for $i,j=1,\cdots,n$. Using Lemma 1, we get (non-zero) orthogonal idempotents $g_1,\cdots,g_m$ in $Q(R)$ satisfying $\sum_{i,j} Q(R)f_{ij}=\sum_i Q(R)g_i$ and $f_{ij}g_k=0$ or g_k for any i,j and k. It follows from $Q(R)=\sum_i Q(R)g_i$ that $1=g_1+\cdots+g_m$. Now let $x\in X(R)$. By Lemma 2, $\left[\left(\sum_i Rg_i\right)_x : R_x\right]\leq 2$, and hence $1_x=(g_i)_x+(g_j)_x$ for some i,j since $\{(g_1)_x,\cdots,(g_m)_x\}$ are linearly independent. We may assume that $1_x=(g_1)_x+(g_2)_x$. Let N be a neighborhood of x such that $1_z=(g_1)_z+(g_2)_z$ for all z in N. Since A is faithful, there must exist a_i and a_j in $\{a_1,\cdots,a_n\}$ such that $a_ig_1\neq 0$ and $a_jg_2\neq 0$. We consider the following two cases in order to find an element a in A such that $a_z\neq 0$ for all z in N. *Case* 1. $a_ig_2\neq 0$. Then it follows that $(*)$ $f_{ij}g_1=g_1$ and $f_{ik}g_2=g_2$ for some j,k. We claim that $(a_i)_z\neq 0$ for all z in N. To see this, suppose $(a_i)_z=0$ for some z in N. Then $(p_{it})_z=(p_{it}(g_1+g_2))_z=0$ and hence $(p_{it}g_1)_z=(p_{it}g_2)_z=0$ for $t=1,\cdots,n$. Hence $(g_1)_z=(g_2)_z=0$ by $(*)$, which shows $1_z=(g_1)_z+(g_2)_z=0_z$, a contradiction. *Case* 2. $a_ig_2=a_jg_1=0$. Then, by a similar argument as above, we can see that $(a_i+a_j)_z\neq 0$ for all z in N. Thus, by Lemma 3, A surely has a basic element.

In the following we shall give an example of a regular ring R such that every finitely generated faithful R-module has a basic element and $X(R)$ contains a 3-point. This implies that the question (1) is "no".

Example. Let R' be the Boolean ring given in Example in [1]. R' has the following conditions: (1′) $X(R')-X_0(R')\neq\emptyset$, where $X_0(R')$ denotes the set of all isolated points in $X(R)$, (2′) there are pairwise disjoint open subsets W_1, W_2, W_3 in $X_0(R')$ such that $X(R')-X_0(R')\subseteq W_i^- - W_i$ for $i=1,2,3$, and (3′) there are orthogonal elements e'_1, e'_2, e'_3 in $Q(R')$ such that $Q(R')=R'e'_1+R'e'_2+R'e'_3$ and for each i, $(e'_i)_z\neq 0$ for all z in W_i. Then we remark that $(e'_j)_z=0$ if z in W_i and $i\neq j$, since $(e'_i)_z\neq 0$ and $[Q(R')_z : R'_z]=1$.

Now let F be a finite field with characteristic$\neq 2$, and consider the simple F-sheaf $X(R')\times F$ over $X(R')$ (see [2, p. 45]). We denote by R the ring of all sections of $X(R')\times F$ over $X(R')$, and by Q the ring of all sections of $X(Q(R'))\times F$ over $X(Q(R'))$. Then Q becomes an extension ring of R by a natural way and $Q=Q(R)=C(R)$ ([1, Proposition 2.3, Theorem 2.4]). Note that $X(R)$ and $X(Q(R))$ are homeomorphic to $X(R')$ and $X(Q(R'))$ respectively. Consequently we get the following properties for R which are parallel to (1′), (2′), (3′) respectively: (1) $X(R)-X_0(R)\neq\emptyset$, (2) there are pairwise disjoint open subsets W_1, W_2, W_3 in $X_0(R)$ such that $X(R)-X_0(R)\subseteq W_i^- - W_i$ for $i=1,2,3$, and (3) there are orthogonal elements e_1, e_2, e_3 in $Q(R)$ such that $Q(R)=Re_1+Re_2+Re_3$ and for each i, $(e_i)_z\neq 0$ for all z in W_i. (Note that $(e_j)_z=0$ if z in W_i and $i\neq j$.)

By (2), $X(R)$ has a 3-point. We now claim that every finitely generated faithful R-module has a basic element. By Lemma 4 and Theorem 5, we may show this for any two-generator faithful non-singular R-module. Let A be a two-generator faithful non-singular R-module, say $A=Ra_1+Ra_2$. By Lemma 3 it is enough to show that for any x in $X(R)$, there is a neighborhood N of x and an element a in A such that $a_z\neq 0$ for any z in N. If x is in $X_0(R)$, then this is clear. So we assume that $x\in X(R)-X_0(R)$. Since A is a two-generator non-singular module, we can assume that A is an R-submodule of $Q(R)\times Q(R)$. Denote each a_i in $Q(R)\times Q(R)$ as $a_i=(r_{i1}e_1+r_{i2}e_2+r_{i3}e_3)\times(s_{i1}e_1+s_{i2}e_2+s_{i3}e_3)$ and put $u_i=(r_{i1}, r_{i2}, r_{i3}, s_{i1}, s_{i2}, s_{i3})$ for $i=1,2$. Consider the R-submodule Ru_1+Ru_2 of $R\times R\times R\times R\times R\times R$. In $(R\times R)_x=R_x\times R_x$, if $(r_{1i}, s_{1i})_x=0$ and $(r_{2i}, s_{2i})_x=0$, then there is a neighborhood N of x for which $(r_{1i}, s_{1i})_z=0$ and $(r_{2i}, s_{2i})_z=0$ for all z in N. Taking z in $W_i\cap N$, it follows that $(e_j)_z=0$ $(j\neq i)$ and $(r_{1i}, s_{1i})_z=(r_{2i}, s_{2i})_z=0$, which show $(a_1)_z=(a_2)_z=0$. Hence $A_z=0$ and so $Ae=0$ for some $0\neq e$ in R, which contradicts the fact that A is faithful. Thus it must hold that $(r_{1i}, s_{1i})_x\neq 0$ or $(r_{2i}, s_{2i})_x\neq 0$ for each i. By making use of this fact we shall find an element $u=(r_1, r_2, r_3, s_1, s_2, s_3)$ in Ru_1+Ru_2 such that $(r_i, s_i)_x\neq 0$, $i=1,2,3$. If $(r_{ji}, s_{ji})_x\neq 0$, $i=1,2,3$. Then we can take u_j as u. If $(r_{11}, s_{11})_x=0$, $(r_{12}, s_{12})_x\neq 0$, $(r_{13}, s_{13})_x\neq 0$, $(r_{21}, s_{21})_x\neq 0$, $(r_{22}, s_{22})_x=0$, $(r_{23}, s_{23})_x=0$, then u_1+u_2 can be choosen as u. In case $(r_{11}, s_{11})_x=0$, $(r_{12}, s_{12})_x\neq 0$, $(r_{13}, s_{13})_x\neq 0$, $(r_{21}, s_{21})_x\neq 0$, $(r_{22}, s_{22})_x=0$, $(r_{23}, s_{23})_x\neq 0$, then, if $(r_{13}, s_{13})_x$ and $(r_{23}, s_{23})_x$ are linearly independent, we can select u_1+u_2 as u. If not, that is, $r_x(r_{13}, s_{13})_x=(r_{23}, s_{23})_x$ for some r in R, then since the characteristic of F is not equal to 2, we can easily take ru_1+u_2 as u.

For such a $u=(r_1, r_2, r_3, s_1, s_2, s_3)$, there exists a neighborhood N of x such that $(r_i, s_i)_z\neq 0$, $i=1,2,3$ for all z in N and the element $a=(r_1e_1+r_2e_2+r_3e_3)\times(s_1e_1+s_2e_2+s_3e_3)$ of A satisfies the condition that $a_z\neq 0$ for any z in N.

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