113. Degenerate Elliptic Systems of Pseudodifferential Equations

By Hideo SOGA
Department of Mathematics, Osaka University

(Communicated by Kôsaku YOSIDA, M. J. A., Oct. 12, 1976)

Introduction. Let M be a compact C^∞ manifold and $A=(a_{ij})_{i,j=1,\cdots,m}$ be a matrix of pseudodifferential operators on M whose symbols, represented by local coordinates, have homogeneous asymptotic expansions (cf. Seeley [4]). Let us consider the equation $Au=f$ on M when A is elliptic outside a C^∞ submanifold M_0 and degenerate on M_0. In the present paper we shall study the normal solvability and the subelliptic estimates for a class of equations such that $\det A_0$ (A_0 is the principal symbol of A) has multi-characteristics, while Èskin in [1] has investigated these problems in the case where $\det A_0$ is of principal type. Finally we shall give an example as an application to non-coercive boundary value problems of fourth order.

1. Assumptions and the main theorem. Let the order of a_{ij} be s_i+t_j ($s_i, t_j \in \boldsymbol{R}$), then A is a continuous operator from $\prod_{j=1}^m H_{s+t_j}(M)$ to $\prod_{i=1}^m H_{s-s_i}(M)$ ($H_s(M)$ denotes the Sobolev space on M of order s). Let M ($n=\dim M \geqq 2$) be separated into two connected components by a C^∞ submanifold M_0. We assume that the ellipticity of A is degenerate on M_0 in the following way.

Let $\{x^i=(x_0^i, \cdots, x_{n-1}^i)\}_{i=1,\cdots,N}$ be a set of local coordinates covering a neighborhood of M_0 and expressing M_0 by the equation $x_0^i=0$, and the transition from x^i to x^j in the domain where both x^i and x^j are defined be given by the form $x_0^j=x_0^i$, $x_k^j=\varphi_k^j(x_1^i, \cdots, x_{n-1}^i)$, $(k=1, \cdots, n-1)$. When A is locally represented in $x^i=(t, y)=(t, y_1, \cdots, y_{n-1})$ $(i=1, \cdots, N)$, its principal symbol $A_0(t, y; \tau, \eta)$ satisfies the assumptions (I)$\sim$(IV):

(I) $\det A_0(t, y; \tau, \eta) \neq 0$ when $t \neq 0$ & $|\tau|+|\eta| \neq 0$ or $t=0$ & $\tau \neq 0$;

(II) $A_0(0, y; 0, \eta)=[0]$ (zero-matrix);

(III) $\det \partial A_0/\partial\tau(0, y; 0, \eta) \neq 0$, $|\eta| \neq 0$;

(IV) Set $\tilde{A}_0(t, y; \eta')=\partial A_0/\partial\tau(t, y; 0, \eta')^{-1}\cdot A_0(t, y; 0, \eta')(\eta'=\eta/|\eta|)$.

There exist positive integers $k_1, \cdots, k_l$ such that the following decomposition of $\tilde{A}_0$ is possible: $t^{-k_1}\tilde{A}_0(t, y; \eta')$ is smooth on $t=0$ and has simple eigenvalues $\lambda_1^1(t, y; \eta'), \cdots, \lambda_{m_1}^1(t, y; \eta')$ with non-vanishing imaginary parts. Other eigenvalues all vanish as $t\to 0$. Let $P_j^1(t, y; \eta')$ be the projection $(2\pi i)^{-1}\oint(\lambda - t^{-k_1}\tilde{A}_0)^{-1}d\lambda$ for the eigenvalue $\lambda_j^1(t, y; \eta')$. Next for $t^{-k_2-k_1}\tilde{A}_0(I-\sum_{j=1}^{m_1}P_j^1)$ the same statements hold. We can continue such a decomposition one after another, and finally have

$$t^{-k_l-k_{l-1}-\cdots-k_1}\tilde{A}_0\Big(I-\sum_{j=1}^{m_1} P_j^1\Big)\cdots\Big(I-\sum_{j=1}^{m_l} P_j^l\Big)=[0]$$

where P_j^i is the projection for the eigenvalue λ_j^i which is obtained in each step. We assume that the above $k_1,\cdots,k_l$ and $m_1,\cdots,m_l$ are all independent of a choice of x^i.

Theorem 1. *If $k_1,\cdots,k_l$ are all even integers, we have*

(i) $\sum_{i=1}^m \|u_i\|_{s+t_i-\varepsilon_0}\leqq C\{\sum_{i=1}^m \|(Au)_i\|_{s-s_i}+\sum_{i=1}^m \|u_i\|_{s+t_i-1}\}$,

$$u=(u_1,\cdots,u_m)\in \prod_{i=1}^m H_{s+t_i}(M) \qquad (s\in \boldsymbol{R})$$

where $\varepsilon_0=(k_1+\cdots+k_l)/(k_1+\cdots+k_l+1)$ and $\|\cdot\|_s$ denotes the norm of $H_s(M)$.

(ii) *The operator $A:\prod_{i=1}^m H_{s+t_i-\varepsilon_0}(M)\to\prod_{i=1}^m H_{s-s_i}(M)$ is of Fredholm type.*

2. An outline of the proof of Theorem 1. It is easily seen that we have only to analyse the symbol $A_0'(x;\xi)=A_0(x;\xi/|\xi|)|\xi|$ $(x=(t,y),\xi=(\tau,\eta))$. We write

$$A_0'(x;\tau,\eta)\theta(\tau,\eta)=A_0'(x;0,\eta)\theta(0,\eta)+A_0^{(1)}(x;\tau,\eta)\tau$$

where $\theta(\xi)\in C^\infty(\boldsymbol{R}^n)$ is equal to 0 if $|\xi|\leqq 1/2$ and to 1 if $|\xi|\geqq 1$.

Lemma 1. i) *For all multi-indices α,β we have*

$$|(\partial/\partial x)^\alpha A_0^{(1)}(x;\xi)|\leqq C_\alpha,\quad \xi\in\boldsymbol{R}^n,\quad x\in U_{\varepsilon_1}=\{(t,y);|t|<\varepsilon_1,|y|<\varepsilon_1\};$$

$$|(\partial/\partial x)^\alpha(\partial/\partial\xi)^\beta A_0^{(1)}(x;\xi)|\leqq C_{\alpha\beta}(1+|\xi|)^{-1},\quad (|\beta|\geqq 1),\quad x\in U_{\varepsilon_1},\quad \xi\in\boldsymbol{R}^n.$$

ii) *When $\varepsilon>0$ is small enough, we have*

$$|\det A_0^{(1)}(t,y;\xi)|\geqq\delta>0,\quad |t|<\varepsilon,\quad (t,y)\in U_{\varepsilon_1},\quad |\xi|\geqq 2.$$

By this lemma it suffices to examine the operator $D_t+\tilde{A}_0(x;D_y/|D_y|)\cdot|D_y|\,\theta(0,D_y)$ $(D_x=(-i\partial/\partial x))$. From the assumption (IV) we can construct the diagonalizer of $\tilde{A}_0$:

Lemma 2. *There exist a finite open covering $\{V_\alpha\}$ on $\{\eta;|\eta|=1\}$ and a set of functions $N_\alpha(x;\eta)\in C^\infty(U_{\varepsilon_1}\times V_\alpha)$ such that for any $(x,\eta)\in U_{\varepsilon_1}\times V_\alpha$* (i) $\det N_\alpha(x;\eta)\neq 0$ and

(ii) $N_\alpha(x;\eta)\tilde{A}_0(x;\eta)$

$$=\begin{bmatrix} t^{k_1}\lambda_1^1(x;\eta) & & & & 0\\ & \ddots & & & \\ & & t^{k_1}\lambda_{m_1}^1(x;\eta) & & \\ & & & \ddots & \\ 0 & & & & t^{k_1+\cdots+k_l}\lambda_{m_l}^l(x;\eta)\end{bmatrix}N_\alpha(x;\eta).$$

Thus we can reduce the problem to the properties of the scalar operator $D_t+t^{2k}\lambda(x;D_y)$, which are well known (cf. [2], [5], etc.).

3. An application to boundary value problems. Let Ω be a bounded open set in $\boldsymbol{R}^n$ $(n\geqq 3)$ with a smooth boundary Γ and Γ be separated into two connected components by a smooth submanifold Γ_0. Suppose that ν_1,ν_2 are real vector fields in $\boldsymbol{R}^n$ defined on Γ of the same type as stated in § 1 of [5] and that the directions of their tangential components to Γ coincide near Γ_0. We assume that $(\nu_i(x), n(x))$ ($n(x)$ is the inner normal unit vector to Γ) converges to zero of order $2k_i$ as dis $(\Gamma_0, x)\to 0$. Let $L(x, D_x)$ be an elliptic differential operator of fourth order on $\bar{\Omega}$ with smooth coefficients. We assume that the equation $L_0(x, \zeta+\omega n(x))=0$ $(x\in\Gamma)$ in ω (L_0 denotes the principal part of L and ζ is any vector $(\neq 0)$ parallel to Γ) has the roots $\omega_1^+(x,\zeta), \omega_2^+(x,\zeta)$ whose imaginary parts are positive and real parts vanish. Let us consider the non-coercive boundary value problem

$$(3.1)\qquad \begin{cases} L(x, D_x)u=f & \text{in } \Omega, \\ \partial^2 u/\partial\nu_2\partial n=g_2 & \text{on } \Gamma, \\ \partial u/\partial\nu_1=g_1 & \text{on } \Gamma. \end{cases}$$

Set $M=\Gamma$, $M_0=\Gamma_0$. Then we see that the Lopatinski matrix (cf. [3]) of (3.1) near Γ_0 satisfies all of the assumptions (I)$\sim$(IV) in some appropriate local coordinates provided that

$$(3.2)\qquad \begin{array}{l} |(\nu_1/|\tilde{\nu}_1|, n)|\leqq|(\nu_2/|\tilde{\nu}_2|, n)| \text{ near } \Gamma_0 \text{ and} \\ \omega_1^+, \omega_2^+ \text{ are distinct on } \Gamma_0 \text{ if } k_1=k_2, \end{array}$$

where $\tilde{\nu}_i$ denotes the component of ν_i parallel to Γ.

Theorem 2. *Let* (3.1) *be coercive outside* Γ_0 *and* (3.2) *hold, then we have for any* $s\geqq 0$

i) $\|u\|_{s+4-\varepsilon_0,\Omega}\leqq C\{\|Lu\|_{s,\Omega}+\|\partial^2 u/\partial\nu_2\partial n\|_{s+3/2,\Gamma}$
$+\|\partial u/\partial\nu_1\|_{s+5/2,\Gamma}+\|u\|_{s+3,\Omega}, \quad u\in H_{s+4}(\Omega),$

where $\varepsilon_0=2k_1/(2k_1+1)$ *and* $\|\cdot\|_{s,\Omega}, \|\cdot\|_{s',\Gamma}$ *denote the norms of* $H_s(\Omega), H_{s'}(\Gamma)$;

ii) *The operator* $u\mapsto(Lu, \partial^2 u/\partial\nu_2\partial n|_\Gamma, \partial u/\partial\nu_1|_\Gamma)$ *is of Fredholm type from* $H_{s+4-\varepsilon_0}(\Omega)$ *to* $H_s(\Omega)\times H_{s+3/2}(\Gamma)\times H_{s+5/2}(\Gamma)$.

References

[1] Èskin, G. I.: Degenerate elliptic pseudodifferential equations of principal type. Math. USSR Sb., **11**, 539–582 (1970).

[2] Kumano-go, H., and Taniguchi, K.: Oscillatory integrals of pseudo-differential operators on R^n and operators of Fredholm type. Proc. Japan Acad., **49**, 397–402 (1973).

[3] Lions, J. L., and Magenes, E.: Non Homogeneous Boundary Value Problems and Applications. I. Springer-Verlag, Berin (1972).

[4] Seeley, R. T.: Topics in pseudo-differential operators. Pseudo-Diff. Operators (C. I. M. E., Stresa 1968), Edizioni Cremonese, Rome, 169–305 (1969).

[5] Soga, H.: Boundary value problems with oblique derivative. Publ. Res. Inst. Math. Sci. Kyoto Univ., **10**, 619–668 (1975).