132. On the Convergence of the Godounov's Scheme for First Order Quasi Linear Equations

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Let $T>0$, $u_0 \in L^\infty(\boldsymbol{R})$, which is assumed of locally bounded variation; we consider the Cauchy's problem:

$$(1)\qquad \frac{\partial u}{\partial t}+\frac{\partial}{\partial x}[f(u,x,t)]+g(u,x,t)=0 \qquad \text{if } (x,t)\in \boldsymbol{R}\times]0,T[;$$

$$(2)\qquad u(x,0)=u_0(x) \qquad \text{if } x\in \boldsymbol{R};$$

where $f\in C^1(\boldsymbol{R}^2\times]0,T[)$, $g\in C^0(\boldsymbol{R}^2\times]0,T[)$ are such that g, f and $\partial f/\partial x$ are Lipschitz continuous with respect to u, uniformly in $(x,t)\in\boldsymbol{R}\times]0,T[$, g and $\partial f/\partial x$ are Lipschitz continuous with respect to x, uniformly in $(u,t)\in\boldsymbol{R}\times]0,T[$, and for $u=0$, $g(0,\cdot,\cdot)$ and $\partial f/\partial x(0,\cdot,\cdot)$ are uniformly bounded on $\boldsymbol{R}\times]0,T[$.

The problem (1), (2) is generally non linear: the solution may be discontinuous and not unique, so we need a weak definition.

Definition 1. A *weak solution* of (1), (2) is a function $u\in L^\infty(\boldsymbol{R}\times]0,T[)$, satisfying:

$$(3)\qquad \iint_{\boldsymbol{R}\times]0,T[}\left\{u\frac{\partial\phi}{\partial t}+f(u,x,t)\frac{\partial\phi}{\partial x}-g(u,x,t)\phi\right\}dxdt+\int_{\boldsymbol{R}}\phi(x,0)u_0(x)dx=0,$$

for any $\phi\in C^2(\boldsymbol{R}\times]0,T[)$, with compact support.

The existence of a weak solution can be proved by the vanishing viscosity method, from the parabolic equation with $\varepsilon>0$:

$$(4)\qquad \frac{\partial u_\varepsilon}{\partial t}+\frac{\partial}{\partial x}[f(u_\varepsilon,x,t)]+g(u_\varepsilon,x,t)=\varepsilon\frac{\partial^2 u_\varepsilon}{\partial x^2},$$

using a compactness argument in $L^1_{\mathrm{loc}}(\boldsymbol{R}\times]0,T[)$ for the family $\{u_\varepsilon\}_{\varepsilon>0}$ (see [3]).

But uniqueness of weak solutions of (1), (2), is not ensured; starting from (4) rather than (1), Kruzkov proposes another definition of solutions, that makes existence and uniqueness sure. See [3], and Hopf [2].

Definition 2. A Kruzkov's solution of (1), (2) is a function $u\in L^\infty(\boldsymbol{R}\times]0,T[)$, satisfying:

$\forall k\in\boldsymbol{R}$, $\forall\phi\in C^2(\boldsymbol{R}\times]0,T[)$, with compact support and non negative:

$$(5)\qquad \begin{aligned}\iint_{\boldsymbol{R}\times[0,T[}\Big\{&|u-k|\frac{\partial\phi}{\partial t}+sg(u-k)(f(u,x,t)-f(k,x,t))\frac{\partial\phi}{\partial x}\\ &-sg(u-k)\left(\frac{\partial f}{\partial x}(k,x,t)+g(u,x,t)\right)\phi\Big\}dxdt\geq 0,\end{aligned}$$

where sg is the sign function: $sg(x)=x/|x|$ if $x\neq 0$, $sg(0)=0$. $\forall R>0\ \exists\mathcal{E}$

$\subset [0, T[$ of measure zero, such that:

$$\lim_{t\to 0, t\notin \mathcal{E}} \int_{|x|<R} |u(x,t)-u_0(x)|\,dx=0. \tag{6}$$

Under certain assumptions of piecewise regularity on u, Hopf [2] proves that it satisfies a well known uniqueness condition, the entropy condition of Oleinik [7]: the good solution is the only one we obtain after replacing $f(\cdot, x, t)$ by its convex (resp. concave) hull at each point $(x, t) \in \boldsymbol{R}\times[0, T[$ on the interval $[u(x-0,t), u(x+0,t)]$ when $u(x-0,t)\le u(x+0,t)$ (resp. $[u(x+0,t), u(x-0,t)]$ when $u(x+0,t)\le u(x-0,t)$).

Taking this condition in account, we get for the solution of the Riemann's problem:

$$\frac{\partial u}{\partial t}+\frac{\partial}{\partial x}[f(u,x,t)]+g(u,x,t)=0\,;\ u(x,0)=\begin{cases} a & \text{if } x<0,\\ b & \text{if } x>0;\end{cases}$$

some information on the line $x=0$, for small t:

$$\begin{aligned}&u(0,t) \text{ is closed to } c\in I(a,b) \text{ such that:}\\ &sg(b-a)f(c,0,0)=\operatorname*{Min}_{k\in I(a,b)}[sg(b-a)f(k,0,0)],\end{aligned} \tag{7}$$

where $I(a, b)$ is the interval $[\operatorname{Inf}(a,b), \operatorname{Sup}(a,b)]$.
Using (7) at each step of the discretization, we can obtain a scheme of order one, the Godounov's scheme, that was described in [6] and [8].

Let $h>0$ be the space meshsize, destinated to vanish. Let $q>0$ be a fixed constant: the time mesh size is taken equal to qh. $\boldsymbol{R}$ and $[0, T[$ are covered by intervals, indexed by $i\in \boldsymbol{Z}$, $n\in\{0,1,\cdots,N=[1+T/qh]\}$

$$I_i=[(i-1/2)h, (i+1/2)h[,$$
$$J_n=[(n-1/2)qh, (n+1/2)qh[\cap[0, T[.$$

Let u be the Kruzkov's solution of (1), (2); u will be approached by a function u_n defined on $\boldsymbol{R}\times[0, T[$, of constant value on each set $I_i\times J_n$, for $i\in\boldsymbol{Z}$, $n\le N$.

We write:

$$u_h(x,t)=u_i^n \qquad \text{if } (x,t)\in I_i\times J_n.$$

The initial condition u_0 is approached on each I_i by the constant:

$$u_i^0=\frac{1}{h}\int_{I_i} u_0(x)dx. \tag{8}$$

We fix $n\le N$, and suppose we know all the constants u_i^n for $i\in\boldsymbol{Z}$; then we construct all the values u_i^{n+1} with the help of the following scheme:

$$\begin{aligned}&u_{i+1/2}^n\in I(u_i^n, u_{i+1}^n)\\ &\text{realized}\quad \operatorname*{Min}_{k\in I(u_i^n,u_{i+1}^n)}[sg(u_{i+1}^n-u_i^n)f(k,(i+1/2)h,nqh)];\end{aligned} \tag{9}$$

$$\begin{aligned}u_i^{n+1}=u_i^n-q[&f(u_{i+1/2}^n,(i+1/2)h,nqh)\\ &-f(u_{i-1/2}^n,(i-1/2)h,nqh)]-qhg(u_i^n, ih, nqh).\end{aligned} \tag{10}$$

The scheme (9), (10) is not exactly the one which is described in [6]. Some differences can appear when $\partial f/\partial u$ presents more than one root; and the convergence to this or that weak solution does depend on the choice of the root (examples are given in [4], [5]), but the value selected by (9) ensures that u_h will converge to the Kruzkov's solution u, when h vanishes. Observe that the minimizing value $u^n_{i+1/2}$ of (9) may be not unique; nevertheless u_i^{n+1} is uniquely determinated for (10) only uses function evaluations with $u^n_{i+1/2}$. We prove the following result:

Theorem. *If the stability condition of Courant-Friedrichs-Lewy:*

$$\forall (w,x,t)\in \boldsymbol{R}^2\times]0,T[\qquad q\left|\frac{\partial f}{\partial u}(w,x,t)\right|\leq 1, \tag{11}$$

is ensured, then the family $\{u_h\}_{h>0}$ *converges in* $L^1_{\mathrm{loc}}(\boldsymbol{R}\times]0,T[)$ *to the Kruzkov's solution of* (1), (2) *when h vanishes.*

Proof. In the following, letters C and M always represent real positive constants, that do not depend on h, i or n. For fixed $h>0$, $n\geq 0$, using (11) we first established the following estimates, the proof of which is very technical and needs a case by case investigation, similar to the methods of [4], [5].

$$\forall i\in \boldsymbol{Z}\ |u_i^{n+1}|\leq (1+C_0h)\ \mathrm{Sup}\,(|u^n_{i-1/2}|, |u^n_i|, |u^n_{i+1/2}|)+C_0'h; \tag{12}$$

$$\forall I\in \boldsymbol{N}\ \sum_{|i|\leq I}|u^{n+1}_{i+1}-u^{n+1}_i|\leq (1+C_1h)\sum_{|i|\leq I+1}|u^n_{i+1}-u^n_i|+C_1'Ih^2; \tag{13}$$

$$\forall I\in \boldsymbol{N}\ \sum_{|i|\leq I}|u^{n+1}_i-u^n_i|\leq \sum_{|i|\leq I+1}|u^n_{i+1}-u^n_i|+C_2Ih(1+\sup_{i\in \boldsymbol{Z}}|u^n_i|); \tag{14}$$

(15) $\forall k\in \boldsymbol{R}, \forall i\in \boldsymbol{Z}$:

$$\begin{aligned}|u_i^{n+1}-k|\leq |u_i^n-k| &- q[sg(u^n_{i+1/2}-k)(f(u^n_{i+1/2},(i+1/2)h,nqh)\\ &\qquad -f(k,(i+1/2)h,nqh))]\\ &+q[sg(u^n_{i-1/2}-k)(f(u^n_{i-1/2},(i-1/2)h,nqh)\\ &\qquad -f(k,(i-1/2)h,nqh))]\\ &-qhsg(u^{n+1}_i-k)[g(u^n_i,ih,nqh)+1/h(f(k,i+1/2)h,ngh)\\ &\qquad -f(k,(i-1/2)h,nqh))]\\ &+ch|u^{n+1}_i-u^n_i|+c'h^2.\end{aligned}$$

We get from (12), step, by step and using (8):

$$\sup_{i\in \boldsymbol{Z}}|u^{n+1}_i|\leq M_0(c_0'/c_0+|u_0|_{L^\infty(\boldsymbol{R})})e^{Tc_0/q} \tag{16}$$

since u_0 is of locally bounded variation, we have:

$$\forall I\in \boldsymbol{N}\ \sum_{|i|\leq I}|u^0_{i+1}-u^0_i|\leq 1/h\int_{|x|\leq Ih}|u_0(x+h)-u_0(x)|\,dx\leq M_0'Ih. \tag{17}$$

Then, from (13), (14) the approximating solution u_h is of locally bounded variation in both variables x and t, uniformly on h. From this and (16), it follows that $\{u_h\}_{h>0}$ is relatively compact in $L^1_{\mathrm{loc}}(\boldsymbol{R}\times]0,T[)$. See [1], [3], [4] and [5] for details on similar methods.

Using the estimates (13), (14), (16), (17) we verify that the limit of any sequence of $\{u_h\}_{h>0}$ is satisfying (6), and starting from (15) with a non negative function $\phi\in C^2(\boldsymbol{R}\times]0,T[)$, with compact support, we obtain (5), by passing through the limit when h vanishes, with regularization arguments to treat the discontinuous term with sign function. Since the Kruzkov's solution is unique, the whole family $\{u_h\}_{h>0}$ will converge to u in $L^1_{\mathrm{loc}}(\boldsymbol{R}\times]0,T[)$, and the theorem is proved.

The scheme (9) (10) generalizes the decentred scheme to the case of a non monotonic function f. It's stable (from (16)) and gives a good representation of discontinuities, without any oscillations, owing to the conservation of the total variation. In [1], Conway and Smoller have established the convergence for the Lax's scheme, using similar arguments; this study was generalized to a wider class of schemes with artificial viscosity terms in [4], [5].

These results can be extended to the p-dimension problem:

$$\frac{\partial u}{\partial t}+\sum_{j=1}^{p}\frac{\partial}{\partial x_j}[f_j(u,x,t)]+g(u,x,t)=0 \qquad \text{if } (x,t)\in \boldsymbol{R}^p\times]0,T[; \tag{18}$$

$$u(x,0)=u_0(x) \qquad \text{if } x\in \boldsymbol{R}^p; \tag{19}$$

that we may numerically solve by the same scheme. We only need to put (18) under the form:

$$\frac{1}{p}\sum_{j=1}^{p}\left\{\frac{\partial u}{\partial t}+\frac{\partial}{\partial x_j}[pf_j(u,x,t)]+g(u,x,t)\right\}=0.$$

Each term of this sum is discretized as an one-dimension problem, with pf_j instead of f_j, that's making harder the stability condition (11), and then their mean value is put equal to zero. We also obtain a convergent family of approached solutions to the Kruzkov's solution of (18), (19), by the use of estimates analogue with (12), (13), (14) and (15). See [3] and [4] more details.

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