# 130. *Factorizations and Fundamental Solutions for Differential Operators of Elliptic-Hyperbolic Type*

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§ 0. **Introduction.** In this note we shall study an operator of the form

(0.1) $$L=D_t^m+A_1(t)D_t^{m-1}+\cdots+A_m(t) \qquad \text{on } [0,T]\times R^n$$

($m\geqq 1, 0<T<\infty$), where $A_j(t)=a_j(t,X,D_x)\in\mathcal{B}_t(\$^j)$ on $[0,T]$ ($j=1,\cdots,m$) (For notations see, for example, Kumano-go [5]). We define the symbol $\sigma(L)=l(t,x,\lambda,\xi)$ for $L$ by

(0.2) $$l=\lambda^m+a_1(t,x,\xi)\lambda^{m-1}+\cdots+a_m(t,x,\xi).$$

We call a symbol $l'=\lambda^m+b_1\lambda^{m-1}+\cdots+b_m$ ($b_j\in\mathcal{B}_t(S^j)$ on $[0,T]$) the principal symbol (or part) of $L$ (or $l$), when we can write $l-l'=\sum_{j=1}^m r_j\lambda^{m-j}$ for $r_j\in\mathcal{B}_t(S^{j-1})$ on $[0,T]$, $j=1,\cdots,m$.

The starting point of the present note is the following factorization theorem, which can be proved by using Sylvester's determinant.

**Theorem 0.** *If the roots $\{\tau_j(t,x,\xi)\}_{j=1}^m$ of $l=0$ are separated into two groups $\{\tau_{1k}\}_{k=1}^{m_1}$ and $\{\tau_{2k}\}_{k=1}^{m_2}$ ($m=m_1+m_2$) so that $|\tau_{1k}-\tau_{2k'}|\geqq C|\xi|$ ($|\xi|\geqq M$) for any $k,k'$ ($C>0, M>0$), then $L$ is factorized into the form*

(0.3) $$L=L_1L_2+\sum_{j=1}^m R_j^{(-\infty)}D_t^{m-j} \qquad \text{on } [0,T]\times R^n$$

*(which is denoted by $L\equiv L_1L_2$ on $[0,T]$), where $R_j^{(-\infty)}\in\mathcal{B}_t(\$^{-\infty})$, and $L_j$ ($j=1,2$) are operators of order $m_j$ such that the principal symbols of $L_j$ are $\prod_{k=1}^{m_j}(\lambda-\tau_{jk}(t,x,\xi))$.*

In § 1 we shall discuss the Levi condition for $L$, and construct the fundamental solution $E(t,s)$, which is represented by Fourier integral operators, when $L$ has the form $L\equiv L^{(+)}L^{(0)}$ (see Theorem 1.3). Then, the Cauchy problem for $L$ can be solved in the spaces $H_s$, $\mathcal{B}$, etc., and the wave front set of the solution can be described through phase functions. Our results are regarded in some sense as global versions of those, obtained by Lax-Nirenberg [8] and Chazarain [1], [2], to $R^n$. We note also that our results can be easily (micro-) localized by considering $aEb$ for appropriate $a, b\in\mathcal{B}_t(\$^0)$ and applying the asymptotic formula for $\sigma(aEb)$ given in [5], which states the canonical relation between $a$ and $b$.

§ 1. **Main theorems.** In what follows we assume that the principal part $l'$ of $l$ has the form $l'=l^{(-)}l^{(+)}l^{(0)}$ where the roots $\{\tau_j^{(\pm)}\}_{j=1}^{m^\pm}$ of $l^{(\pm)}=0$

satisfy $\operatorname{Im}\tau_j^{(\pm)} \gtrless C|\xi|$ $(|\xi|\geqq M)$ for $C>0$, $M>0$, and $l^{(0)}$ has the form $l^{(0)} = \prod_{j=1}^{r}(\lambda-\tau_j^{(0)})^{\nu_j}$ $(m^0=\nu_1+\cdots+\nu_r)$ for real valued $\tau_j^{(0)} \in \mathscr{B}_t(S^1)$ on $[0, T]$ such that $|\tau_j^{(0)}-\tau_{j'}^{(0)}|\geqq C|\xi|$ $(|\xi|\geqq M,\ j\neq j')$. Then, by means of Theorem 0, there exist operators $L^{(\pm)}$, $L^{(0)}$ and $L_j^{(0)}$ of order $m^{\pm}$, $m^0$ and $\nu_j$, whose principal symbols are $l^{(\pm)}$, $l^{(0)}$ and $(\lambda-\tau_j^{(0)})^{\nu_j}$, respectively, such that $L$ has the form

(1.1) $$L\equiv L^{(-)}L^{(+)}L^{(0)}\equiv L^{(-)}L^{(+)}L_r^{(0)}\cdots L_1^{(0)} \quad \text{on } [0, T].$$

Now let $\{\phi_j(t,s;x,\xi)\}_{j=1}^{r}$ be phase functions which satisfy

(1.2) $$\partial_t\phi_j-\tau_j^{(0)}(t,x,\nabla_x\phi_j)=0 \quad \text{on } [s, T_s],\ \phi_j|_{t=s}=x\cdot\xi.$$

Such $\phi_j$ exist for $T_s=\operatorname{Min}\{s+\delta_0, T\}$ with some $\delta_0>0$, and satisfy conditions: "i) $\phi_j-x\cdot\xi \in \mathscr{B}_t(S^1)$, ii) $|\nabla_x\phi_j-\xi|\leqq(1-\varepsilon_0)|\xi|+C_0$, iii) $\|\nabla_x\nabla_\xi\phi_j-I\|\leqq(1-\varepsilon_0)$ for $0<\varepsilon_0\leqq 1$, $C_0>0$ (see [5])".

Then, in the analogy of Chazarain [1], [2] (c.f. also [3], [7]) we give

**Definition 1.1.** We say that $L$ satisfies the Levi condition (denoted by "$(L-C)$") on $[0, T]\times R^n$, if for any $s\in[0, T]$ and any $a\in\mathscr{B}_t(S^k)$ on $[s, T_s]$ we have

(1.3) $$e^{-i\phi_j}L(e^{i\phi_j}a)\in\mathscr{B}_t(S^{m-\nu_j+k}) \quad \text{on } [s, T_s],\ j=1,\cdots,r.$$

**Theorem 1.2.** i) *$L$ satisfies $(L-C)$ on $[0, T]\times R^n$ if and only if each $L_j^{(0)}$ satisfies $(L-C)$ on $[0, T]\times R^n$.* ii) *$L_j^{(0)}$ satisfies $(L-C)$ on $[0, T]\times R^n$ if and only if there exist $R_{j,k}\in\mathscr{B}_t(\$^0)$ on $[0, T]$, $k=1,\cdots,\nu_j$, such that for $Q_j=D_t-\tau_j^{(0)}(t,X,D_x)$ we have*

(1.4) $$L_j^{(0)}=Q_j^{\nu_j}+R_{j,1}Q_j^{\nu_j-1}+\cdots+R_{j,\nu_j} \quad \text{on } [0, T].$$

Now we assume that $L$ has the form as the special case

(1.5) $$L\equiv L^{(+)}L^{(0)}\equiv L^{(+)}L_r^{(0)}\cdots L_1^{(0)} \quad \text{on } [0, T]$$

and consider the Cauchy problem for $\varphi=(\varphi_1,\cdots,\varphi_m)^t$

(1.6) $$Lu=0 \quad \text{on } [s, T_s],\ D_t^{j-1}u|_{t=s}=\varphi_j\ (j=1,\cdots,m).$$

We write $l^{(+)}=\lambda^{m^+}+b_1^{(+)}\lambda^{m^+-1}+\cdots+b_{m^+}^{(+)}$ and set

$$\begin{aligned}
&U=(u_1,\cdots,u_m)^t,\ F=(0,\cdots,0,f)^t,\ u_k=Q_1^{k-1}u\ (1\leqq k\leqq\nu_1),\\
&\quad =Q_j^{k-\bar\nu_{j-1}-1}L_{j-1}^{(0)}u_{\bar\nu_{j-2}+1}\ (2\leqq j\leqq r,\ \bar\nu_{j-1}+1\leqq k\leqq\bar\nu_j),\\
&\quad =\Lambda^{m-k}D_t^{k-m^0-1}L_r^{(0)}u_{\bar\nu_{r-1}}\ (m^0+1\leqq k\leqq m),\\
&\qquad\qquad (m=m^++m^0,\ \bar\nu_j=\nu_1+\cdots+\nu_j,\ \bar\nu_0=0),
\end{aligned}$$

$$A=\begin{bmatrix} A_1^{(0)} & & & 0\\ & \ddots & & \\ & & A_r^{(0)} & \\ 0 & & & A^{(+)}\end{bmatrix},\quad A_j^{(0)}=\begin{bmatrix}\tau_j^{(0)}(t,X,D_x) & & 0\\ & \ddots & \\ 0 & \nu_j & \tau_j^{(0)}(t,X,D_x)\end{bmatrix},$$

$$A^{(+)}=\begin{bmatrix} 0 & \Lambda & \cdots & 0\\ & \ddots & \ddots & \\ 0 & & 0 & \Lambda\\ -\Gamma_m & \cdots & \cdots & -\Gamma_1\end{bmatrix},\quad \Gamma_j=b_j^{(+)}(t,X,D_x)\Lambda^{-(j-1)} \quad (\sigma(\Lambda)=(1+|\xi|^2)^{1/2}),$$

$$B=(B_{j,k};\ j,k=1,\cdots,r+1),\qquad B_{j,j}=\begin{pmatrix} 0 & 1 & \cdots & 0\\ & & \ddots & 1\\ -R_{j,\nu_j} & \cdots & \cdots & -R_{j,1}\end{pmatrix},$$

$$B_{j,j+1}=\begin{pmatrix}0 & \cdots 0\\ & \cdots\\ 1 & 0\cdots 0\end{pmatrix}\ (1\leqq j\leqq r),\qquad B_{j,k}=0\ \text{(otherwise)}.$$

Then, corresponding to (1.6) we have

$$(1.6)'\qquad LU=D_tU-AU-BU=F\quad \text{on } [s,T_s],\ U|_{t=s}=\psi,$$

where $\psi=(\psi_1,\cdots,\psi_m)^t=G(s)\varphi$ for some

$$G(s)=\begin{pmatrix}1 & & 0\\ & \ddots & \\ g_{jk}(s) & & 1\end{pmatrix}\qquad (g_{jk}(t)\in\mathscr{B}_t(s^{j-k})\ \text{on } [0,T],\ j>k).$$

By the similar way to the method constructing the perfect diagonalizer in [6] (see also [10]) there exists $N\in\mathscr{B}_t(\$^0)$ on $[0,T]$ of the form

$$N\sim I+\sum_{\mu=1}^{\infty}N^{(-\mu)},\ N^{(-\mu)}=\left(n_{j,k}^{(-\mu)}\in\mathscr{B}_t(\$^{-\infty})\,;\ n_{j,k}^{(-\mu)}=0\right.$$

except $n^{(-\mu)}_{p_j-k,p_j+1+k'}$ $\begin{pmatrix}1\leqq j\leqq r-1\\ 0\leqq k+k'\leqq\mu-1\end{pmatrix}$ and $n^{(-\mu)}_{m^0-k,m^0+k'}$ $\left.\begin{pmatrix}0\leqq k\leqq\mu-1\\ 0\leqq k'\leqq m^+\end{pmatrix}\right)$, such that $LN\equiv NL_0$ on $[0,T]$, where

$$L_0U=D_tU-AU-B_0U\qquad \text{for } B_0=\begin{pmatrix}B_{11,} & 0 & \\ & B_{r,r} & \\ 0 & & 0\end{pmatrix}.$$

Then, applying the results in [6] and [9] (or [4], [11]) for $D_t-A_j^{(0)}$ and $D_t-A^{(+)}$, respectively, we can construct for $L_0$ the approximate fundamental solution in the sense of [6]

$$\tilde{E}_0(t,s)=\begin{pmatrix}\tilde{E}_{\phi_1}(t,s) & & & 0\\ & \ddots & & \\ & & \tilde{E}_{\phi_r}(t,s) & \\ 0 & & & \tilde{E}_{\phi^+}(t,s)\end{pmatrix}\in\mathscr{B}_t(\$^0)\qquad \text{on } [s,T_s]\ (\phi^+=x\cdot\xi),$$

i.e., $L_0\tilde{E}_0\in\mathscr{B}_t(\$^{-\infty})$ on $[s,T_s]$ and $\tilde{E}_0|_{t=s}=I$. Consequently, by solving an integral equation with a pseudo-differential operator as the kernel (see [4], [11]), we have

**Theorem 1.3.** *For the problem* (1.6) *there exists the fundamental solution* $E(t,s)=(E_1(t,s),\cdots,E_m(t,s))$ $(0\leqq s\leqq t\leqq T_s\leqq T)$, *which has the form*: $E(t,s)=$"*the first row of* $N(t)\tilde{E}_0(t,s)N_{-1}(s)G(s)$" + "$E^{(-\infty)}(t,s)$" *for some* $E^{(-\infty)}\in\mathscr{B}_t(\$^{-\infty})$ *on* $[s,T_s]$ *and satisfies*

$$LE_j(t,s)=0\quad on\ [s,T_s],\qquad D_t^{l-1}E_j|_{t=s}=\delta_{j,l}\ (j,l=1,\cdots,m),$$

*where* $N_{-1}(t)$ ($\in\mathscr{B}_t(\$^0)$ *on* $[0,T]$) *is the parametrix of* $N(t)$ *which has the form* $\sigma(N_{-1}(t))\sim I-\sum_{\mu=1}^{\infty}(-1)^{\mu}\sigma((N(t)-I)^{\mu})$ *on* $[0,T]$.

The detailed descriptions will be published elsewhere.

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