146. *On Algebraic Threefolds of Parabolic Type*

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§ 1. In the present note all algebraic varieties are assumed to be complete, irreducible and defined over the complex number field $\boldsymbol{C}$. A non-singular algebraic variety is called an algebraic manifold.

Let V be an algebraic manifold and we let K_V (resp. Ω_V^p) denote the canonical bundle (resp. the sheaf of germs of holomorphic p-forms) of V. Put

$$P_m(V)=\dim_C H^0(V, \mathcal{O}(mK_V)), \qquad m=1,2,3,\cdots,$$
$$h^{p,0}(V)=\dim_C H^0(V, \Omega_V^p), \qquad p=1,2,3,\cdots,\dim V.$$

It is well-known that these are birational invariants. Further we put

$$p_g(V)=P_1(V),$$
$$q(V)=h^{1,0}(V).$$

$p_g(V)$ (resp. $q(V)$) is called the geometric genus (resp. the irregularity) of V. For a singular algebraic variety V we define

$$p_g(V)=p_g(V^*),$$
$$q(V)=q(V^*),$$

where V^* is a non-singular model of V.

If $P_m(V)$ is positive for a natural number m, we define a rational mapping (the m-th canonical mapping)

$$\begin{array}{rccl} \Phi_{mK}: & V & \longrightarrow & \boldsymbol{P}^N \\ & \cup\!\!\!\text{ı} & & \cup\!\!\!\text{ı} \\ & z & \longmapsto & (\varphi_0(z):\varphi_1(z):\cdots:\varphi_N(z)) \end{array}$$

where $\{\varphi_0, \varphi_1, \cdots, \varphi_N\}$ is a basis of $H^0(V, \mathcal{O}(mK_V))$. We set $N(V) =\{m>0 \mid P_m(V)>0\}$. The *Kodaira dimension* $\kappa(V)$ of an algebraic manifold V is defined by

$$\kappa(V)=\begin{cases} \max\limits_{m\in N(V)} \dim \Phi_{mK}(V) & \text{if } N(V)\neq\emptyset, \\ -\infty & \text{if } N(V)=\emptyset. \end{cases}$$

The Kodaira dimension $\kappa(V)$ is a birational invariant. Therefore, for a singular algebraic variety V we define

$$\kappa(V)=\kappa(V^*),$$

where V^* is a non-singular model of V.

An algebraic manifold V is called *parabolic type* if $\kappa(V)=0$. This is equivalent to saying that $P_m(V)\leqq 1$ for every positive integer m and there exists a positive integer n such that $P_n(V)=1$.

In the present note we shall give an outline of a proof of the following theorem. The details will be published elsewhere.

Theorem. *Let V be a three-dimensional algebraic manifold V of parabolic type. That is, $P_m(V) \leqq 1$ for every positive integer m and $P_n(V)=1$ for some n. Then we have the following.*

1) *The Albanese mapping $\alpha: V \rightarrow A(V)$ is surjective, where $A(V)$ is the Albanese variety of V. Hence, a fortiori, $q(V) \leqq 3$. Moreover, if $p_g(V)=1$, then $q(V) \neq 2$.*

2) *$q(V)=3$ if and only if V is birationally equivalent to an abelian variety of dimension three.*

3) *If $q(V)=1$, a general fibre of the Albanese mapping α is an abelian surface or a K3 surface. In the former case, there exists a finite unramified covering $\tilde{V}$ of V which is birationally equivalent to the product of an abelian surface and an elliptic curve.*

This is an affirmative answer to Problems 2, 10 and 11 concerning algebraic manifolds of parabolic type raised by Iitake [1] under the assumption that the dimension is three. (See also Conjectures Q_3, A_3, B_3 in Ueno [2] p. 129–131.)

The following corollary is an immediate consequence of the theorem.

Corollary. *An algebraic threefold V is birationally equivalent to an abelian variety of dimension three, if and only if*

$$P_m(V) \leqq 1, \quad m=1,2,3,\cdots, P_n(V)=1 \qquad \text{for some } n>0$$
$$q(V)=3.$$

§2. To prove the theorem we need several lemmas.

Lemma 1. *Let $\varphi: V \rightarrow C$ be a surjective morphism of an n-dimensional algebraic manifold V to a non-singular curve C of genus $g \geqslant 2$ with connected fibres. Suppose $\chi(V) \geq 0$. Then we have*

$$\kappa(V) \geq 1.$$

Moreover if V is of dimension three, we have

$$\kappa(V) \geq \kappa(V_x)+1$$

where V_x is a general fibre of φ.

Lemma 2. *Let $\varphi: V \rightarrow A$ be a surjective morphism of a three-dimensional algebraic manifold V onto a two-dimensional abelian variety A. Suppose that a general fibre of φ is a non-singular elliptic curve. Then $\kappa(V) \geq 0$. Moreover, $\kappa(V)=0$, if and only if $\varphi: V \rightarrow A$ is birationally equivalent to a fibre bundle over A in the sense of etale topology whose fibre is an elliptic curve.*

Lemma 3. *Let $\varphi: V \rightarrow E$ be a surjective morphism of a three-dimensional algebraic manifold V onto a non-singular elliptic curve E. Suppose that a general fibre of φ is an abelian variety of dimension two. Then $\kappa(V) \geq 0$. Moreover, $\kappa(V)=0$, if and only if $\varphi: V \rightarrow E$ is* *birationally equivalent to a fibre bundle over E in the sense of etale topology whose fibre is an abelian variety of dimension two.*

Lemma 4. *Let V be an n-dimensional algebraic manifold with*

$$p_g(V)=1,$$
$$q(V)=n.$$

Suppose that the Albanese mapping $\alpha: V \to A(V)$ is surjective. Then α induces isomorphisms

$$\alpha^*: H^0(A(V), \Omega_A^k) \xrightarrow{\sim} H^0(V, \Omega_V^k), \qquad k=1,2,\cdots,n.$$

§3. Using Lemma 1, the main theorem of Viehweg [3] and Corollary 10.6 in Ueno [2], we can show that the Albanese mapping $\alpha: V \to A(V)$ is surjective if $\chi(V)=0$. By virtue of Lemma 2, it is not difficult to show that if $p_g(V)=1$, then $q(V)\neq 2$.

Next we consider the second part of the theorem. We let $\sum n_i S_i$ be the *effective* canonical divisor. Set $\mathcal{S}=\bigcup S_i$. It is enough to show that $\alpha(\mathcal{S})$ is *not* of codimension one. Moreover, if the Albanese variety $A(V)$ is not simple (that is, $A(V)$ contains an elliptic curve), we can prove the theorem. Therefore, we can assume that $\alpha(S_1)$ is of codimension one and $A(V)$ is simple. Further we can assume that $S=S_1$ is non-singular. Then, by Corollary 10.10 in Ueno [2], $\alpha(S)$ is a surface of general type, hence a fortiori, so is S. Thus we obtain

$$p_g(S)-q(S)+1>0$$
$$q(S)\geq q(\alpha(S))\geq 3.$$

On the other hand, by Lemma 4, we have

$$p_g(S)\leq h^{2,0}(V)=3.$$

Hence $\chi(S, \mathcal{O}_S)=p_g(S)-q(S)+1=1$. Considering a suitable finite unramified covering $\tilde{V}$ of V ($\tilde{V}$ also satisfies the assumption of the second part of the theorem), we show that the equality $\chi(S, \mathcal{O}_S)=1$ implies a contradiction. This proves the second part.

Finally Lemma 3 implies the third part of the theorem.

References

[1] S. Iitaka: Genera and classification of algebraic varieties. I (in Japanese). Sugaku, **24**, 14–27 (1972).

[2] K. Ueno: Classification theory of algebraic varieties and compact complex spaces. Lecture Notes in Math., **439** (1975). Springer-Verlag.

[3] E. Viehweg: Canonical divisors and the additivity of the Kodaira dimension for morphisms of relative dimension one (to apear in Composition Math.).