34. *On the Periods of Enriques Surfaces. I*

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§1. **Introduction.** A non-singular algebraic surface S is called an Enriques surface if the following two conditions are satisfied:

(i) The geometric genus and the irregularity both vanish.

(ii) If K is a canonical divisor on S, $2K$ is linearly equivalent to 0.

Historically speaking Enriques surfaces were the first example of non-rational algebraic surfaces which satisfy the above condition (i). In this paper we are mainly interested in Enriques surfaces over the field of complex numbers $\boldsymbol{C}$.

From the condition (ii), it follows that there exists a two-sheeted unramified covering $\pi: T \to S$ such that T is a $K3$ surface. Since every $K3$ surface is known to be simply-connected by Kodaira [6], T is the universal covering of S. We take a holomorphic 2-form ψ on T which is non-zero everywhere, and consider the integrals

$$(1) \qquad \int_\gamma \psi \qquad \text{for } \gamma \in H_2(T, \boldsymbol{Z}).$$

We let τ denote the covering transformation $T \to T$ over S so that $\tau^2 = \mathrm{id}$. Since S has no holomorphic 2-form, we have $\tau^*\psi = -\psi$. On the other hand, τ acts on $H_2(T, \boldsymbol{Z})$ as an involution. If γ is invariant by τ, then the above integral (1) vanishes. Therefore the periods of ψ are determined by the integrals (1) over those 2-cycles γ satisfying $\tau\gamma = -\gamma$. Our main result is that the isomorphism class of S is uniquely determined by these periods. A more precise statement will be given in §4. Details will be published elsewhere.

§2. **Elliptic surfaces of index 2.** It is known that an Enriques surface S has a structure of an elliptic surface (see [1], [8]). That is, there exists a surjective holomorphic map $g: S \to \boldsymbol{P}^1$ whose general fibre C is an elliptic curve. Moreover there exists a divisor G on S with $CG = 2$. From Kodaira's formula for the canonical bundles of elliptic surfaces ([6], p. 772), it follows that g has two multiple fibres, both being of multiplicity 2. We view S as an elliptic curve over the function field of $\boldsymbol{P}^1$. Then G is a divisor of degree 2 on this curve. Hence G defines a rational map f_1 of degree 2 of S onto a rational ruled surface W_1. This map induces, for each generic fibre C, a double covering $C \to \boldsymbol{P}^1$ which is ramified at 4 points. Let B_1 be the branch locus of f_1. By applying elementary transformations defined in [3] we can modify f_1 into a similar rational map $f: S \to W$ onto a rational ruled surface W such that its branch locus B has only the following singularities:

(i) At most a simple triple point, that is, without infinitely near triple points (see [2]).

(ii) B contains a fibre Γ and $B_0 = B - \Gamma$ has a double point s on Γ which, on performing a quadratic transformation at s, gives a double point of its proper transform on the proper transform of Γ.

A singularity of the second type corresponds to a double fibre. Hence in our case B has exactly two of them. Here we remark that the choice of W is not unique because, if we apply an elementary transformation at s as in (ii), we obtain another ruled surface and the new branch locus still satisfies the above condition. Because of this phenomenon, we may always take $\boldsymbol{P}^1 \times \boldsymbol{P}^1$ as W. In this way we obtain a birational model of S which is a double covering of $\boldsymbol{P}^1 \times \boldsymbol{P}^1$. This model is closely related to the model studied in [1] and [8], which is a double covering of $\boldsymbol{P}^2$.

§3. Two propositions. The construction in §2 proves the following

Proposition 1. *Any two Enriques surfaces are deformation to each other.*

To state the second proposition we recall that, if T is a $K3$ surface, $H_2(T, \boldsymbol{Z})$ is an even unimodular euclidean lattice of signature (3, 19). Hence it is isomorphic to

$$\Lambda = U_1 \oplus U_2 \oplus U_3 \oplus E_8 \oplus E_8',$$

where $U_i = \boldsymbol{Z}x_i + \boldsymbol{Z}y_i$ with $x_i y_i = 1$, $x_i^2 = y_i^2 = 0$ $(i = 1, 2, 3)$ and E_8, E_8' are copies of the unique even unimodular negative-definite lattice of rank 8. We define an involution $\rho: \Lambda \to \Lambda$ by the conditions $\rho | U_1 = -id$, $\rho(x_2) = x_3$, $\rho(y_2) = y_3$, $\rho(E_8) = E_8'$ and that ρ induces the identity $E_8 \to E_8'$. We fix (Λ, ρ) once and for all.

Proposition 2. *Let T be the universal covering of an Enriques surface S. Then there exists an isomorphism of euclidean lattices $\varphi: H_2(T, \boldsymbol{Z}) \to \Lambda$ which satisfies $\varphi \circ \tau = \rho \circ \varphi$, where τ denotes the involution on $H_2(T, \boldsymbol{Z})$ induced by the covering transformation.*

By Proposition 1, it suffices to prove Proposition 2 for one special S. On the other hand, if $g: S \to \boldsymbol{P}^1$ has a singular fibre of type II* (see Kodaira [5], p. 565), i.e., a singular fibre which has the configuration of the extended Dynkin diagram of type $\tilde{E}_8$, then it is easy to prove the proposition for S. Therefore the proof is reduced to constructing an Enriques surface with a singular fibre of type II*. This can be done by using the construction described in §2.

§4. **Main Theorem.** (Λ, ρ) being as above, we let $\Lambda(-1)$ denote the (-1)-eigenspace of ρ. Then $\Lambda(-1)$ is a euclidean lattice of signature $(2, 10)$ with determinant 2^{10}. Let S be an Enriques surface and let $\varphi: H_2(T, \boldsymbol{Z}) \to \Lambda$ be as in Proposition 2. Then the integrals (1) determines, via φ, a linear map $\omega: \Lambda(-1) \to \boldsymbol{C}$ which satisfies the Riemann bilinear relation (see [7]). Hence ω can be viewed as a point of an open set D in a quadric in $\boldsymbol{P}^{11}$. D is a disjoint union of two copies of a 10-dimensional symmetric bounded domain of type IV. We let Γ' denote the group of those automorphisms of Λ which commute with the involution ρ. Then Γ' induces a group Γ of automorphism of $\Lambda(-1)$, and Γ acts discontinuously on D. The image $\lambda(S)$ of ω on D/Γ is uniquely determined by S and does not depend on the choice of φ. We call $\lambda(S)$ the *period* of S.

Main Theorem. *The isomorphism class of an Enriques surface S is uniquely determined by its period $\lambda(S) \in D/\Gamma$.*

The proof uses the Torelli theorem for $K3$ surfaces due to Piateckii-Shapiro and Shafarevich [7]. Suppose that two Enriques surfaces S_1 and S_2 have the same period. Then, if $\pi_i: T_i \to S_i$ $(i=1,2)$ are the universal coverings, there exists an isomorphism $\varphi: H_2(T_1, \boldsymbol{Z}) \to H_2(T_2, \boldsymbol{Z})$ which is compatible with involutions and preserves periods. If φ maps effective cycles into effective cycles, then φ is induced by a unique isomorphism $\Phi: T_1 \to T_2$, and Φ is compatible with involutions. Hence S_1 and S_2 are isomorphic to each other. If φ does not preserve effective cycles, then we can compose φ with a reflexion

$$\gamma \longrightarrow \gamma + (\gamma \cdot \pi_2^* e)\pi_2^* e$$

with respect to the class e of a rational curve on S_2. Composing φ with a finite number of such reflexions we may assume that either φ or $-\varphi$ preserves effective cycles, and then we are through as above.

Remarks. 1) From the explicit description of (Λ, ρ), it follows that D/Γ is connected.

2) It can also be proved that Γ is an arithmetic subgroup of $SO(2, 10)$ with respect to the $\boldsymbol{Q}$-structure defined by $\Lambda(-1)$.

3) It is very likely that our method in [4] can be applied to study the image of the period map λ for Enriques surfaces.

4) Our construction in §2 also works over any algebraically closed field of characteristic $\neq 2$. Using this construction we can prove that any Enriques surface in characteristic $\neq 2$ can be lifted to characteristic 0.

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