33. A Note on Malmquist's Theorem on First Order Algebraic Differential Equations

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The entitled theorem reads as follows: *If the differential equation*

(1) $dw/dz=R(z,w)$ (*R is a rational function of z and w*)

has a transcendental meromorphic solution $w(z)$, then the equation must be of the Riccati type, i.e., $R(z,w)$ must be a polynomial of the second degree in w.

In 1933 the present author gave, as an application of the Nevanlinna theory ([5]) of meromorphic functions, another proof of this striking theorem of J. Malmquist [4] dating 1913. In this proof (Yosida [9] and [10]), a decisive role was played by a theorem of G. Valiron [6]:

(2) $$T(r, R(z, w(z)))^{1)}=d\cdot T(r, w(z))+O(\log r),$$

where d is the degree in w of $R(z,w)$. In 1950, H. Wittich ([7] and [8]) gave an alternate proof which is based upon the fact that the order of the meromorphic function $w(z)$ is finite and that its proximity function $m(r, w(z))$ is $O(\log r)$. Recently in 1974, E. Hille ([2] and [3]) gave another approach proposing a geometric argument instead of Wittich's estimation via the calculus of residues. It is to be noted here that, for the finiteness of the order of the meromorphic solution $w(z)$, the author gave in 1934 a straightforward proof ([10], Theorem 7) relying upon the T. Shimizu-L. Ahlfors-H. Cartan interpretation (see, e.g., [5], 165–) of the Nevanlinna characteristic $T(r, w(z))$.

In view of the above, I should like to show that my original idea in [9] and [10] can be pursued to the result without appealing to the theorem of Valiron nor to the Wittich-Hille type estimation.

We may assume that

(3) $$R(z,w)=P(z,w)/Q(z,w)=\left(\sum_{j=0}^{p} p_j(z)w^j\right)/\left(\sum_{k=0}^{q} q_k(z)w^k\right)$$

with polynomial coefficients p_j's and q_k's such that $p_p(z)\cdot q_q(z)\not\equiv 0$ and w-polynomials $P(z,w)$ and $Q(z,w)$ have no factos in common. By virtue of the *defect relation* in the Nevanlinna theory, we have

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1) We shall follow notations in [1]:
$T(r,f(z))=m(r,f(z))+N(r,f(z))$, $m(r,f(z))=(2\pi)^{-1}\int_0^{2\pi}\log^+|f(re^{i\theta})|\,d\theta$, $N(r,f(z)) =\int_0^t t^{-1}(n(t,f(z))-n(0,f(z)))dt+n(0,f(z))\cdot\log r$, where $n(r,(f(z))$ denotes the number of poles of $f(z)$ for $|z|\leqq r$, multiple poles being counted with the multiplicity.

$$\overline{\lim_{r\uparrow\infty}} N\left(r, \frac{1}{w(z)-C}\right)\Big/ T(r, w(z))=1 \tag{4}$$

except possibly for countable sequence of complex numbers C's. Hence there exist a complex number C_0 and an increasing sequence of positive numbers r_j's with $r_j \uparrow \infty$ such that

$$\lim_{j\to\infty} N\left(r_j, \frac{1}{w(z)-C_0}\right)\Big/ T(r_j, w(z))=1 \tag{4$'$}$$

and

$$P(z, C_0)\not\equiv 0 \quad \text{and} \quad Q(z, C_0)\not\equiv 0. \tag{5}$$

Thus, by the first fundamental theorem of the Nevanlinna theory, we have

$$\lim_{j\to\infty} m\left(r_j, \frac{1}{w(z)-C_0}\right)\Big/ T(r_j, w(z))=0. \tag{6}$$

Next, by the transformation

$$W(z)=\frac{1}{w(z)-C_0} \tag{7}$$

we obtain, from (1) and (3),

$$dW/dz=P_1(z, W)/Q_1(z, W) \tag{8}$$

where the degrees in W of $P_1(z, W)$ and of $Q_1(z, W)$ are

$$\begin{cases} p \text{ and } p-2, \text{ respectively when } p-2\geqq q \\ \text{or} \\ q+2 \text{ and } q, \text{ respectively when } q>p-2. \end{cases} \tag{9}$$

Hence we have

$$dW/dz=a_2(z)W^2+a_1(z)W+a_0(z)+(P_2(z, W)/Q_1(z, W)) \tag{10}$$

where $a_j(z)$'s are rational functions with $a_2(z)\not\equiv 0$ and the degree $\hat{p}_2$ in W of $P_2(z, W)$ satisfies

$$\begin{cases} \hat{p}_2\leqq(p-2)-1 \qquad \text{when } (p-2)\geqq q, \\ \text{or} \\ \hat{p}_2\leqq q-1 \qquad \text{when } q>(p-2). \end{cases} \tag{11}$$

Therefore the theorem of Malmquist is proved if we can show that

$$P_2(z, W)/Q_1(z, W) \text{ does not contain } W. \tag{12}$$

The Proof of (12). Assume the contrary. Then (11) implies that

$$\begin{cases} P_2(z, W(z))/Q_1(z, W(z))=0 \text{ at every pole of } W(z) \text{ except} \\ \text{possibly for finite number of } z\text{'s}. \end{cases} \tag{13}$$

Moreover,

$$P_2(z, W(z))/Q_1(z, W(z))=W'(z)-a_2(z)W(z)^2-a_1(z)W(z)-a_0(z) \tag{14}$$

takes finite value for all z at which $W(z)$ is finite, except possibly for finite number of z's. Therefore (as in Wittich [8] and Hille [3])

$$N(r, P_2(z, W(z))/Q_1(z, W(z)))=O(\log r). \tag{15}$$

Again by (14), we obtain (as in Wittich [8] and Hille [3])

$$\begin{cases} m(r, P_2(z, W(z))/Q_1(z, W(z))) \\ \qquad \leqq m(r, W'(z))+2m(r, W(z))+m(r, W(z))+O(\log r) \\ \qquad \leqq m(r, W(z))+m(r, W'(z)/W(z))+3m(r, W(z))+O(\log r). \end{cases} \tag{16}$$

On the other hand, as was stated in the beginning of the present note, we obtain

(17) $T(r, W(z))=O(r^k)$ with a positive integer k

by using to *the Shimizu-Ahlfors-Cartan interpretation of the Nevanlinna characteristic* $T(r, W(z))$:

$$(17)' \quad \begin{cases} \text{for any } r_0>0 \text{ and any non-void open set } G \text{ of the complex plane,} \\ T(r, W(z))=O\left(\int_{r_0}^{r} s^{-1}ds\left[\iint_{D(G,r_0\leqq t\leqq s)} \frac{|W'(te^{i\theta})|^2}{(1+|W(te^{i\theta})|^2)^2} tdtd\theta\right]\right), \\ \text{where} \\ D(G, r_0\leqq t\leqq s) \text{ denotes the set } \{te^{i\theta};\, W(te^{i\theta})\in G,\ r_0\leqq t\leqq s\}. \end{cases}$$

In fact, substituting (8) in (17)′ and taking $r_0>0$ and G appropriately, we obtain (17). The finiteness of the order of $W(z)$, expressed in (17), implies $m(r, W'(z)/W(z))=O(\log(r\cdot T(r, W(z)))$ ([1], Theorem 2.2) and so

(18) $T(r, P_2(z, W(z))/Q_1(z, W(z)))=O(\log(r\cdot T(r, W(z)))+O(m(r, W(z))$

by (15) and (16). Moreover, we have, by (13) and the first fundamental theorem of the Nevanlinna theory,

$$\begin{aligned} T(r_j, P_2(z, W(z))/Q_1(z, W(z)) &\geqq N(r_j, Q_1(z, W(z))/P_2(z, W(z)))+O(1) \\ &\geqq N(r_j, W(z))+O(\log r_j) \qquad (\text{as } j\to\infty). \end{aligned}$$

This contradicts to (18) because of (4)′, (6), $T(r, W(z))=T(r, w(z))+O(1)$ and the fact that $\log(r\cdot T(r, W(z)))=o(T(r, W(z))$ which is implied by the transcendental meromorphic assumption of $W(z)$.

We have thus proved (12) and so the proof of the theorem of Malmquist is completed.

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