121. *Note on the Conformal Representation.*

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Let $V_\nu(z)$, $(\nu=0,1,2,\cdots)$ be a set of regular analytic functions, which form a complete system of normalized orthogonal functions on a simply closed analytic curve C of length l:

$$\frac{1}{l}\int_C V_\mu(z)\overline{V_\nu(z)}\,ds\begin{cases}=0 & \text{for } \mu\neq\nu,\\ =1 & \text{for } \mu=\nu.\end{cases}$$

Then the series

$$\sum_{\nu=0}^{\infty} V_\nu(z)\,\overline{V_\nu(a)},\quad (a \text{ in } C)$$

is convergent absolutely and uniformly in the closed region interior to C, and represents a definite function $K(z,a)$ dependent only on the curve C.

Now let $\{f(z)\}$ be a set of functions, regular and analytic in C, such that

$$\frac{1}{l}\int_C |f(z)|^p\,ds \leqq 1,\quad (p>0).$$

Of these functions that which makes $|f(\zeta)|$ (ζ in C) a maximum is

$$f^*(z)=\varepsilon_1\left\{\frac{K(z,\zeta)^2}{K(\zeta,\zeta)}\right\}^{\frac{1}{p}},\qquad (|\varepsilon_1|=1)^{1)}$$

This problem may also be solved by the conformal transformation.

Let $x=\chi(z,a)$ be the equation by which the interior of C is transformed conformally into the interior of the unit circle about the origin of the x-plane, the point a corresponding to the origin, and let $z=\omega(x,a)$ be the inverse representation.

1) S. TAKENAKA, General mean modulus of analytic functions, Tôhoku Math. Journal, 27 (1926).

Then we have

$$\frac{1}{l}\int_{c}|f(z)|^{p}\,ds=\frac{1}{2\pi}\int_{0}^{2\pi}\left|f(\omega(x,a))\left\{\frac{2\pi}{l}\,\frac{\partial\omega(x,a)}{\partial x}\right\}^{\frac{1}{p}}\right|^{p}d\theta\leqq 1,\quad (x=e^{i\theta}).$$

Since, under the condition

$$\frac{1}{2\pi}\int_{0}^{2\pi}|\varphi(x)|^{p}\,d\theta\leqq 1,\qquad (x=e^{i\theta}),$$

the function which is regular and analytic for $|x|<1$ and makes $|\varphi(t)|$, $(|t|<1)$ a maximum is

$$\varepsilon_{2}\left\{\frac{1-|t|^{2}}{(1-\bar{t}x)^{2}}\right\}^{\frac{1}{p}},\qquad (|\varepsilon_{2}|=1)^{1)},$$

we have, putting $t=\chi(\zeta,a)$,

$$f^{*}(\omega(x,a))\cdot\left\{\frac{2\pi}{l}\,\frac{\partial\omega(x,a)}{\partial x}\right\}=\varepsilon_{3}\left\{\frac{1-|\chi(\zeta,a)|^{2}}{(1-x\,\overline{\chi(\zeta,a)})^{2}}\right\}^{\frac{1}{p}},\quad (|\varepsilon_{3}|=1),$$

so that we obtain[2)]

$$(2)\qquad f^{*}(z)=\varepsilon_{3}\left\{\frac{}{2\pi}\,\frac{\partial\chi(\cdot,a)}{\partial z}\right\}\quad\left\{\frac{1-|\chi(\zeta,a)|^{2}}{1(-\chi(z,a)\,\overline{\chi(\zeta,a)})^{2}}\right\}^{\frac{1}{p}},\quad (|\varepsilon_{3}|=1),$$

or, in particular, if we put $\zeta=a$, we get

$$(3)\qquad f^{*}(z)=\varepsilon_{3}\left\{\frac{l}{2\pi}\,\frac{\partial\chi(z,a)}{\partial z}\right\}^{\frac{1}{p}},\qquad (|\varepsilon_{3}|=1).$$

Now comparing (1) with (2), it follows that

$$\frac{K(z,\zeta)}{K(\zeta,\zeta)^{\frac{1}{2}}}=\varepsilon\left\{\frac{l}{2\pi}\,\frac{\partial\chi(z,a)}{\partial z}\right\}^{\frac{1}{2}}\frac{(1-|\chi(\zeta,a)|^{2})^{\frac{1}{2}}}{1-\chi(z,a)\,\overline{\chi(\zeta,a)}},\quad (|\varepsilon|=1)$$

Equating the conjugate values of both sides and putting $z=\zeta$, we have, since $K(z,\zeta)=\overline{K(\zeta,z)}$,

$$K(\zeta,\zeta)^{\frac{1}{2}}=\bar{\varepsilon}\left\{\frac{l}{2\pi}\,\frac{\partial\chi(\zeta,a)}{\partial\zeta}\right\}^{\frac{1}{2}}\frac{1}{(1-|\chi(\zeta,a)|^{2})^{\frac{1}{2}}}\,.$$

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1) This may be easily proved by the conformal transformation $\xi=\dfrac{x-t}{1-\bar{t}x}$.

2) Remembering that $\dfrac{\partial\chi}{\partial z}\cdot\dfrac{\partial\omega}{\partial x}=1$ for the corresponding values of z and x.

Hence it follows that

$$(4)\qquad K(z,\zeta)=\frac{l}{2\pi}\left\{\frac{\partial\chi(z,a)}{\partial z}\cdot\overline{\frac{\partial\chi(\zeta,a)}{\partial\zeta}}\right\}^{\frac{1}{2}}\frac{1}{1-\chi(z,a)\overline{\chi(\zeta,a)}},$$

or, in particular,

$$(5)\qquad K(z,a)=\frac{l}{2\pi}\left\{\frac{\partial\chi(z,a)}{\partial z}\right\}^{\frac{1}{2}}\left\{\overline{\frac{\partial\chi(z,a)}{\partial z}}\right\}^{\frac{1}{2}}_{z=a}$$

Thus we see that *the definite function* $K(z,a)$ *may be expressed by* (4) *or* (5). These formulas have been obtained by other method in my paper: "On some properties of orthogonal functions etc.," to appear in the Japanese Journal of Math., **3** (1926).