118. *On a Generalization of Picard's Theorem.*

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Landau has shown that, if $f(x) = a_0 + a_1x + a_2x^2 + \cdots\cdots$ is a transcendental integral function, where $a_0 \neq 0, 1$, $a_1 \neq 0$, then $f(x)$ takes the value 0 or 1 in a circle $|x| \leqq R(a_0, a_1)$, where R depends only on a_0 and a_1. The following theorem gives some more information about *the distribution of 0 and 1-points of $f(x)$ outside this circle.*

Theorem. Let $f(x) = a_0 + a_1x + \cdots\cdots$ be an integral function, where $a_0 \neq 0$ and $f(x_\nu) = 0$, $(0 < |x_1| < |x_2| < \cdots \to \infty)$ and a and b be given constants, then there exists a sequence of circles $|x| = R_\nu$, $(0 < R_1 < R_2 \cdots \to \infty)$ such that in any ring region $R_\nu < |x| \leqq R_{\nu+1}$ $(R_0 = 0,\ \nu = 0, 1 \cdots)$, $f(x)$ takes the value a or b; the radii of circles $|x| = R_\nu$ $(\nu = 1, 2 \cdots)$ depending only on a_0, x_ν $(\nu = 1, 2 \cdots)$ and a, b.

The condition imposed on $f(x)$ requires only that it should vanish at x_ν; the multiplicity of zero is arbitrary, and $f(x)$ may vanish at other points than x_ν.

Lemma. Under the condition of the theorem, when a circle $|x| = R$ is given, we can find a second circle $|x| = R'$ $(R < R')$, so that $f(x)$ takes the value a or b in the ring region $R < |x| \leqq R'$, where R' depends only on a_0, x_ν $(\nu = 1, 2 \cdots)$, a, b and R.

Suppose that the lemma is false, then we can find a sequence of circles $|x| = R_\nu$ $(R < R_1 < R_2 \cdots \to \infty)$ and functions $f_\nu(x)$ so that $f_\nu(x)$ does not take the values a and b in the ring region $R < |x| \leqq R_\nu$, where $f(x)$ satisfies the condition of our Theorem.

Since $f_1(x)$, $f_2(x)$, $\cdots\cdots$ do not take the values a and b in $R < |x| \leqq R_1$, they form a normal family, so that we can select a subsequence $f_{11}(x), f_{12}(x), \cdots\cdots$, which converge uniformly in $R < |x| < R_1$. Since $f_{12}(x)$, $f_{13}(x), \cdots\cdots$ do not take the values a and b in the ring region $R < |x| \leqq R_2$, we can select a sub-sequence $f_{22}(x)$, $f_{23}(x) \cdots\cdots$ which converge uniformly in $R < |x| < R_2$, and so on. Thus we get a sequence $f_{11}(x)$, $f_{22}(x)$, $f_{33}(x), \cdots\cdots$ which converge uniformly in $R < |x| < R'$, where R' is any large number such that in $R < |x| < R'$ there exists at least one x_ν.

They can not converge uniformly to infinity, since $f_{nn}(x_\nu)=0$; hence they converge to a regular function. By Weierstrass's theorem they converge uniformly in the circle $|x|<R'$, to a limiting function $f(x)$, which, since R' is arbitrary, is an integral function.

On the other hand, $f(x)$ is not a constant, since $f(0)=a_0\neq 0$, $f(x_\nu)=0$; and $f(x)$ is not a polynomial, since it vanishes at infinitely many points x_ν. Hence $f(x)$ is a transcendental integral function and it does not take the values a and b outside the circle $|x|=R$, in contradiction to the theorem of Picard. Thus the lemms is proved.

Proof of the Theorem. By the lemma we can find a circle $|x|=R_1$, in which $f(x)$ takes the value a or b, and then the second circle $|x|=R_2$, so that $f(x)$ takes the value a or b in the ring region $R_1<|x|\leqq R_2$, and so on. In this way we obtain a sequence of circles $|x|=R_\nu$, such that $\lim_{n=\infty} R_n=\infty$, since, if not, there must exist a circle $|x|=R'$, so that $R_n<R'$, $(n=1, 2\cdots)$, and in which $f(x)$ takes the value a or b infinitely many times, contradictory to the hypothesis that $f(x)$ is an integral function. Thus the theorem is proved.

Remark. Several extention of the theorem may be made; for example, instead of giving a_0 itself, we may suppose only $|a_0|\geqq k_0>0$, where k_0 is given.