7. *Asymmetric Vibrations.*

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Asymmetric vibrations, which are often found in motions of imperfectly elastic bodies, was first treated by Helmholtz[1] in his theory of combination tones, and by many physicists with special relation to acoustics,[2] optics and thermal expansion.[3] The solution is mostly approximative, as the imperfection of elastic property is supposed to be very small, but a more complete solution is sometimes desirable, when the deviation from Hooke's law is considerable.

The general aspect of the problem can be presented in the following form. Suppose a particle to be acted upon by an attractive force F(r) depending on distance r. If it be displaced by ξ

$$F(r+\xi) = F(r)+\xi\frac{\partial F}{\partial r}+\frac{\xi^2}{2}\frac{\partial^2 F}{\partial r^2}+\cdots\cdots \qquad (1)$$

If the restitutive force be simply proportional to the displacement, the vibration is simple harmonic, but if $\xi^2\frac{\partial^2 F}{\partial r^2}$ be not negligible, the vibration becomes asymmetric. In imperfectly elastic bodies, we have to retain ξ^2 and higher powers, especially when the amplitude is not small; a most remarkable example is presented by rock materials constituting the earth's crust. In addition to this, they show plasticity, which is of great importance in many problems of geophysics.

To simplify the calculation, consider a particle displaced by ξ from its position of equilibrium; then the equation of motion is

$$m\frac{d\xi}{dt^2} = -f\xi+g\xi^2, \qquad (2)$$

retaining only the first two terms of the restitutive force; f and g are constants measured by the differential coefficients of (1). The first integral leads to

$$\frac{m}{2}\left(\frac{d\xi}{dt}\right)^2 = a-f\frac{\xi^2}{2}+g\frac{\xi^3}{3} \qquad (3)$$

where a denotes the initial energy. By the introduction of damping force proportional to velocity, a simply substitution leads to a cubic of similar form. As the problem is to find the effect of imperfect elasticity, the damping is taken out of consideration.

The mathematical part of the problem is to integrate

$$\frac{d\xi}{\sqrt{a-\frac{1}{2}f\xi^2+\frac{1}{3}g\xi^3}} = \sqrt{\frac{2}{m}}dt \tag{4}$$

From the physical nature of the problem, we know that $f\xi$ is much greater than $g\xi^2$, so that to first approximation, we can neglect $g\xi^2$ and obtain simple harmonic motion, as was done by previous investigators; and by the process of successive approximation arrive at the solution of the problem. This way of procedure does not well reveal the dependence of the period on factor g, introduced in the above equation. On finding the roots of the equation

$$a-\frac{1}{2}f\xi^2+\frac{1}{3}g\xi^3 = 0, \tag{5}$$

we see that two of them are given approximately by $\pm\sqrt{\frac{2a}{f}}$. Putting

$$\xi = \sqrt{\frac{2a}{f}}\, y,$$

equation (5) becomes $\quad 1-y^2+\alpha y^3 = 0,$

where $\quad \alpha = \frac{2}{3}\frac{g}{f}\sqrt{\frac{2a}{f}}$

Thus $\quad y = \pm\sqrt{1+\alpha y^3}$

By successive approximation, we find

$$y_2 = 1+\frac{1}{2}\alpha+\frac{5}{8}\alpha^2+\alpha^3+\cdots\cdots \tag{6}$$

and

$$y_3 = -1+\frac{1}{2}\alpha-\frac{5}{8}\alpha^2+\alpha^3-\cdots\cdots; \tag{6'}$$

for y_1, we have the relation $y_1+y_2+y_3=\frac{1}{\alpha}$,

whence

$$y_1=\frac{1}{\alpha}-\alpha-2\alpha^3+\cdots\cdots \tag{6''}$$

Thus ξ_1, ξ_2, ξ_3 are found.

Evidently $k = \frac{1}{2}\alpha+\frac{19}{16}\alpha^{23}+\cdots\cdots$, and since $\left(\frac{1-k}{1+k}\right)^2 = \frac{\xi_1-\xi_2}{\xi_1-\xi_3}$, (7) we put $$\xi = \frac{\xi_2+\xi_3}{2} - \frac{\xi_3-\xi_2}{2}\frac{1}{k} - \frac{\xi_3-\xi_2}{2}\frac{k'^2}{k}\frac{1}{(1-kz)} \tag{8}$$

The integral to be found is reduced to the canonical form

$$\int\frac{d\xi}{\sqrt{a-\frac{1}{2}f\xi^2+\frac{1}{3}g\xi^3}} = \frac{1}{\mu}\int\frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}, \tag{9}$$

where $$\frac{1}{\mu} = 2\sqrt{\frac{3}{g}}\sqrt{\frac{k}{\xi_2-\xi_3}} = \sqrt{\frac{2}{f}}\left(1+\frac{7}{8}\alpha^2+\cdots\cdots\right) \tag{9'}$$

Consequently $$z = \text{sn}\left(\mu\sqrt{\frac{2}{m}}t+\varepsilon\right) \tag{10}$$

The period of vibration is given by $$T = \frac{2\sqrt{2}}{\mu}K, \tag{11}$$

K being complete elliptic integral of the first kind. On expansion in terms of known constants a, f and g,

$$\begin{aligned} T &= 2\pi\sqrt{\frac{m}{f}}\left(1+\frac{5}{6}\frac{g^2a}{f^3}+\cdots\cdots\right) \qquad (11') \\ &= \left(1+\frac{5}{6}\frac{g^2a}{f^3}\right)T_0, \end{aligned}$$

since k is generally small.

By expanding (8) in powers of z, and expressing sn and its powers in trigonometrical functions, we find for the displacement ξ, by putting $\omega = \frac{2\pi}{T}$, and choosing the origin of time properly,

$$\begin{aligned} \xi = \frac{ag}{f^2} - \sqrt{\frac{2a}{f}}\left(1+\frac{37}{12}\frac{ag^2}{f^3}\right)\sin\omega t + \frac{1}{3}\frac{ag}{f^2}\cos 2\omega t \\ + \frac{1}{26}\sqrt{\frac{2a}{f}}\frac{ag^2}{f^3}\sin 3\omega t+\cdots\cdots \qquad (12^*) \end{aligned}$$

The period of asymmetric vibration here sketehed is *prolonged by an amount proportional to initial energy a*, and also *proportional to the square of the constant g*, measuring the imperfection of elasticity, and *inversely proportional to the cube of f*. The prolongation is considerable for a particle with slow period, as the most effective part is played by the smallness of f. Consequently we have to expect the presence of fluctuations in the period for particles bound imperfectly elastic and vibrating slowly. The proper period may vary according to the change

* These calculations were checked by Mr. S. Sakurai of the Institute.

in g and a. In seismic waves passing through the crust of the earth, these two constants are important factors, and we have on this account to look for fluctuations, which though small may sometimes be appreciable.

Another important consequence is the *permanent set*, which may be observable and given by the first term in equation (12). The amplitude of vibration is also affected by g, which has the tendency to increase it. The presence of g gives rise to vibrations which are the harmonics of the principal. Owing to the smallness of g, the second harmonic will be generally inappreciable, but the first considerable, especially in soft orcks, such as are found near the surface of the earth. According to Kusakabe's[4] investigation, the hysteresis is of great amount with torsional strain, so that the *said effects are probably found in transverse or S waves.* Prof. A. Imamura was kind enough to show me seismograms obtained by explosion; the component vibrations transverse to the direction of propagation show period, which is *half* of that in the longitudinal component. Mr. S. Kunitomi, of the Central Meteorological Observatory, showed me a number of seismograms, obtained with Wiechert, Mainka, Galitzin and Omori's instruments, indicating the presence of *octaves in the transverse components* of many earthquakes, but some doubts are not yet cleared, if the vibrations so recorded are not to be attributed to instrumental defects.

When a particle is acted upon by harmonic forces of frequencies p and q, the presence of combination frequencies $\omega + mp + nq$ is naturally to be expected. This was formerly tested by Kusakabe[5] on specimens of rocks, who found that m and n can become so large as 6. In seismic waves, the presence of such combination frequencies may be found on close examination of the seismograms, but as several vibrations are superimposed we are still in lack of our knowledge on this important point. Perhaps we can trace their existence *in pulsatory oscillations*, as the motion seems to take place in thin layer of surface rocks, in which the elastic nature is very defective.

The treatment of the problem when the moving point is affected with hysteresis, so that the equation is to be changed on forward and backward motion, is more intricate. In case of necessity of taking further term ξ^3 into account, the calculation is rendered somewhat complicated, but the mode of reduction for effecting the integration is similar; we can by proper substitution reduce the biquadratic form and proceed in the same manner as here indicated. For the full treatment of the problem on the propagation of waves in an imperfectly elastic medium, we have to take the squares of the strain components into account. The subject, so far as I am aware, has only been slightly touched.

References.

1) Helmholtz: Wissenschaftliche Abhandl. Bd. 1 (1882) 256.
2) Rayleigh: Theory of Sound, Vol. 1 (1894) 76.
3) Debye: Wolfskehl Vortrag, Göttingen (1913); Born: Atomtheorie des festen Zustandes, (1923) 67; Mogendorff: Proc. Akad. Weten. (1911) 470.
4) Kusakabe: Publ. Earthq. Inv. Comm. No. 11, 1903; Journ. Coll. Sci. 19 (1903) Art. 6.
5) Nagaoka; Proc. Phys. Math. Soc. 2 (1905) 443.