32. *Note on a Theorem of Fekete.*

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1. Fekete[1] and Bálint[2] proved the following theorem:

If

$$P(z) = p_0+p_1z^{\mu_1}+p_2z^{\mu_2}+\cdots\cdots\cdots\cdots\cdots+p_kz^{\mu_k}$$

be a polynomial with $k+1$ terms ($p_0, p_1 \cdots\cdots\cdots, p_k$ are any complex numbers other than zero; and $\mu_1, \mu_2, \cdots\cdots\cdots, \mu_k$ are integers such that $1 \leqq \mu_1 < \mu_2 < \cdots\cdots < \mu_k$), and $P(-1) \neq P(+1)$, then there exists at least one point z in the circle $|z| \leqq 2 \cdot k \cot \frac{\Phi}{2} \left(\Phi \leqq \frac{\pi}{2}\right)$ in which $P(z)$ takes any given value γ in the domain K', whose boundary consists of two circular arcs subtending an angle Φ to the segment joining the points $P(-1)$ and $P(+1)$.

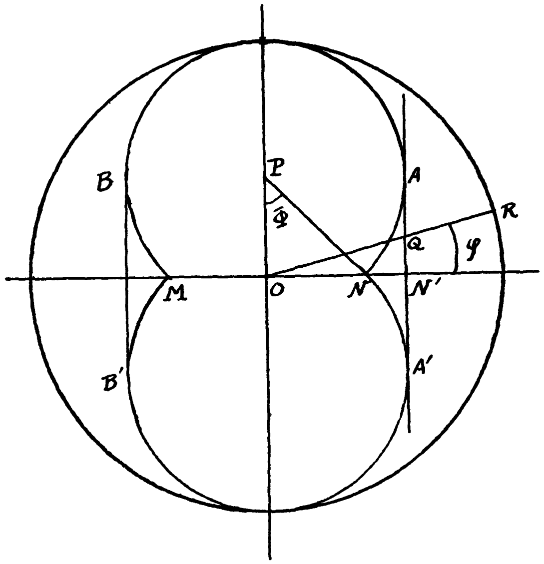
We can, however, extend this domain for γ into the circle K with centre $\{P(-1)+P(+1)\}/2$ and radius $\left\{|P(+1)-P(-1)| \cot \frac{\Phi}{2}\right\}/2$, which contains K'.

Our theorem runs as follows:

Theorem 1. *Let $P(-1) \neq P(+1)$, and γ be any point in the circle K with centre $\{P(-1)+P(+1)\}/2$ and radius $\frac{1}{2}|P(+1)-P(-1)| \cot \frac{\Phi}{2}$, where $\Phi \leqq \frac{\pi}{2}$. Then there exists at least one point z in the circle $|z| \leqq 2\,k \cot \frac{\Phi}{2}$, in which $P(z)$ takes the value γ.*

Proof. Draw two circular arcs passing through the points $P(-1)$, $P(+1)$, subtending an angle $\Phi \leqq \frac{\pi}{2}$. Let AA', BB' be the common tangents of two circles and O the midpoint of $M(P(-1))\ N(P(+1))$. Take a point Q on AA' and a point $R(\gamma)$ on the line OQ. Then since we have

1) Fekete, Jahrsb. d. Deutsch. Math. Ver. 32 (1923), 299-306.
2) Bálint, The same Journal, 34 (1926), 233-237.



$$\overline{OQ} = \overline{ON'}/\cos\varphi = \overline{PN}/\cos\varphi = \overline{ON}/\{\sin\Phi\cos\varphi\}, \qquad (1)$$

putting

$$\overline{OR}/\overline{OQ} = \lambda,\ 2\overline{ON}\cdot e^{i\alpha} = |P(+1)-P(-1)|\cdot e^{i\alpha}$$
$$= P(+1)-P(-1), \qquad (2)$$

we get $\gamma = \{P(+1)+P(-1)\}/2+\overline{OR}\, e^{i(\varphi+\alpha)}$

$= \{P(+1)+P(-1)\}/2+[\lambda e^{i\varphi}\{P(+1)-P(-1)\}]/\{2\sin\Phi\cos\varphi\}$,

i.e. $\gamma = \sigma P(-1)+\tau P(+1), \qquad (3)$

where $\sigma = \dfrac{1}{2}\left\{1-\dfrac{\lambda e^{i\varphi}}{\sin\Phi\cos\varphi}\right\},\ \tau = \dfrac{1}{2}\left\{1+\dfrac{\lambda e^{i\varphi}}{\sin\Phi\cos\varphi}\right\}$,

whence $\tau+\sigma = 1,\ |\tau-\sigma| = \dfrac{\lambda}{\sin\Phi\cos\varphi}. \qquad (4)$

Now consider the locus of R for which $\dfrac{\lambda}{\sin\Phi\cos\varphi} = \cot\dfrac{\Phi}{2}$, which reduces, by means of (1) and (2), to the relation $\overline{OR} = \lambda\cdot\overline{OQ} = \overline{ON}\cot\dfrac{\Phi}{2}$, or

$$OR = \left|\frac{P(+1)-P(-1)}{2}\right|\cdot\cot\frac{\Phi}{2}. \qquad (5)$$

That is, the locus of R is the circle K, mentioned in the theorem, which touches obviously the above circular arcs.

Thus for γ in K or on the boundary, we have

$$|\tau-\sigma| \leqq \cot\frac{\Phi}{2}. \tag{6}$$

From (3), we get

$$(\sigma+\tau)p_0-\gamma+p_1r_1+p_2r_2+\cdots\cdots\cdots+p_kr_k = 0,$$

$$r_s = (-1)^{\mu_s}\sigma+\tau, \quad s = 1,2,3,\cdots\cdots\cdots, k.$$

$$|p_0-\gamma| \leqq \cot\frac{\Phi}{2}\cdot(|p_1|+|p_2|+|p_3|+\cdots\cdots\cdots+|p_k|),$$

since $|r_s| = |(-1)^{\mu_s}\sigma+\tau| \leqq \cot\frac{\Phi}{2}$.

Therefore there exists an integer $s \leqq k$, for which $|p_0-\gamma| \leqq 2^s|p_s| \cot\frac{\Phi}{2}$,[1]

whence
$$\left|\frac{p_0-\gamma}{p_s}\right|^{\frac{1}{\mu_s}} \leqq 2^{\frac{s}{\mu_s}}\left(\cot\frac{\Phi}{2}\right)^{\frac{1}{\mu_s}} \leqq 2\cot\frac{\Phi}{2}. \tag{7}$$

Then it follows[2] that the equation

$$P(z)-\gamma = p_0-\gamma+p_1z^{\mu_1}+p_2z^{\mu_2}+\cdots\cdots\cdots+p_kz^{\mu_k} = 0$$

has at least one root in the circle $|z| \leqq r$, where

$$r \leqq k\left|\frac{p_0-\gamma}{p_s}\right|^{\frac{1}{\mu_s}} \leqq 2k\cot\frac{\Phi}{2}.$$

Thus the theorem is proved.

2. Next we can prove the following

Theorem 2. *Let* $\gamma = \sigma P(-1)+\tau P(+1)$, *where*

$$\sigma+\tau = 1, \quad |\tau-\sigma| \leqq M. \tag{8}$$

Then there exists at least one point z *in the circle* $|z| \leqq 2\cdot Mk$ *for which* $P(z)$ *takes any value* γ^* *in the circle* K_1 *(inclusive of the boundary) with centre* $\{P(-1)+P(+1)\}/2$ *and radius* $|\gamma-\{(P(-1)+P(+1))\}/2|$.

Proof. By the hypothesis, we can find for any γ^* in K_1, λ and φ, such that

$$\gamma^* = \frac{P(-1)+P(+1)}{2}+\lambda\left\{\gamma-\frac{P(-1)+P(+1)}{2}\right\}e^{i\varphi}, \ (0 \leqq \lambda \leqq 1).$$

That is
$$\gamma^* = \sigma^*P(-1)+\tau^*P(+1), \tag{9}$$

where
$$\sigma^* = \frac{1}{2}+\sigma\lambda e^{i\varphi}-\frac{1}{2}\lambda e^{i\varphi}, \quad \tau^* = \frac{1}{2}+\tau\lambda e^{i\varphi}-\frac{1}{2}\lambda e^{i\varphi},$$

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1) Fekete, loc. cit. 303.

2) Fekete, loc. cit. Hilfsatz V, 300–301.

whence $\sigma^* + \tau^* = 1, \quad |\tau^* - \sigma^*| \leqq M.$

Hence we can prove our theorem by a similar way as the last part of the proof of Theorem 1.

From this theorem we can deduce Theorem 1; for, we may take γ lying collinear with the points $P(-1)$, $P(+1)$ so that $\sigma > 0$, $\tau < 0$. Then from the relations $\tau + \sigma = 1$, $\sigma - \tau = M$, we get $\sigma = \{M-1\}/2$, $\tau = -\{M-1\}/2$, whence

$$\left|\gamma - \frac{P(-1)+P(+1)}{2}\right| = M \cdot \left|\frac{P(+1)-P(-1)}{2}\right|.$$

Hence putting $M = \cot\frac{\Phi}{2}$, we get Theorem 1.

3. Finally we can extend these results to power series:

Theorem 3. *If* $f(z) = p_0 + p_1 z^{\mu_1} + p_2 z^{\mu_2} + \cdots + p_k z^{\mu_k} + \cdots\cdots$ *be a transcendental integral function, for which the series* $\frac{1}{\mu_1} + \frac{1}{\mu_2} + \cdots\cdots$ *converges, and* $f(-1) \neq f(+1)$, *then* $f(z)$ *takes any value in the circle* K *in the Theorem 1 for at least one point* z *in the circle* $|z| \leqq 8 \exp\left\{\sum_{k=2}^{\infty} \frac{1}{\mu_k - 1}\right\} \cdot \cot\frac{\Phi}{2}$.

Similarly the theorem corresponding to Theorem 2 can be easily seen.

In conclusion I express my cordial thanks to Prof. Y. Okada for his kind suggestion.