31. *On the Power Series Whose Initial Coefficients are Given.*

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Let C_ν, $(\nu = 0,1,\cdots,n)$ be given constants not all zero (without any loss of generality let us suppose that $C_0 \neq 0$); and consider a set $\{f(z)\}$ of functions, regular and analytic for $|z| < 1$, and such that

$$f(z) \equiv \sum_{\nu=0}^{n} C_\nu z^\nu, \quad (\text{mod. } z^{n+1}).$$

Of these functions that which makes the integral

$$I(f) = \frac{1}{2\pi}\int_{|z|=1} |f(z)| \cdot |dz|$$

minimum is a rational integral function $f^*(z)$ of a degree not exceeding $2n$. This result has been proved by F. RIESZ[1].

In this note I will give the inferior limit of $I(f)$ and the true expression of $f^*(z)$ when C_ν $(\nu = 0,1,\cdots,n)$ satisfy certain conditions.

Let $\varphi(z)$ be an arbitrary regular function of the form

$$\varphi(z) = \sum_{\nu=0}^{\infty} a_\nu z^\nu, \qquad (|z| < 1),$$

under the condition that

$$|\varphi(z)| \leqq M \quad \text{for } |z| < 1.$$

Then it can easily be seen that

$$(1) \qquad \left|\sum_{\nu=0}^{n} C_\nu a_{n-\nu}\right| \leqq \frac{M}{2\pi}\int_{|z|=1} |f(z)| \cdot |dz|.$$

Now put

$$(x_0 + x_1 z + \cdots + x_n z^n)^2 \equiv \sum_{\nu=0}^{n} C_\nu z^\nu, \ (\text{mod. } z^{n+1}),$$

1) F. RIESZ, Ueber Potenzreihen mit vorgeschriebenen Anfangsgliedern, Acta Math., 42 (1920), 145.

so that

$$(2) \qquad C_\mu = \sum_{\nu=0}^{\mu} x_{\mu-\nu} x_\nu, \qquad (\mu = 0,1,\cdots,n),$$

and let $a_\nu (\nu = 0,1,\ldots,n)$ be the solution of the following linear algebraic equations:

$$(3) \qquad \begin{cases} x_n a_0 + x_{n-1} a_1 + \cdots + x_0 a_n = \varepsilon \bar{x}_0, \\ x_{n-1} a_0 + x_{n-2} a_1 + \cdots + x_0 a_{n-1} = \varepsilon \bar{x}_1, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_1 a_0 + x_0 a_1 = \varepsilon \bar{x}_{n-1}, \\ x_0 a_0 = \varepsilon \bar{x}_n. \end{cases} \qquad (|\varepsilon| = 1).$$

Then by a well-known theorem of CARATHÉODORY and FEJÉR, we can uniquely determine a rational function $\varphi^*(z)$ of a degree not exceeding n, regular for $|z| < 1$ and of constant absolute magnitude m $(m > 0)$ for $|z| = 1$, and such that

$$\varphi^*(z) \equiv \sum_{\nu=0}^{n} a_\nu z^\nu, \qquad (\text{mod. } z^{n+1}),$$

where m^2 is the greatest characteristic value of non-negative Hermitian form $H(x,\bar{x})$ whose matrix is

$$H = \bar{S}S, \; S = \begin{pmatrix} & & a_0 \\ & a_0, & a_1 \\ & \cdots\cdots\cdots & \\ a_0, & \cdots\cdots, & a_{n-1} \\ a_0, a_1, & \cdots\cdots, & a_n \end{pmatrix}$$

If we take this function $\varphi^*(z)$ instead of $\varphi(z)$ in (1), then, by the relations (2) and (3), it can easily be seen that the equality (1) takes the form

$$(4) \qquad \sum_{\nu=0}^{n} |x_\nu|^2 \leqq \frac{m}{2\pi} \int_{|z|=1} |f(z)| \cdot |dz|,$$

from which we see that *the inferior limit of $I(f)$ is* $\frac{1}{m} \sum_{\nu=0}^{n} |x_\nu|^2$.

If therefore we can find a function which satisfies the equality in (4), then it should be the required function $f^*(z)$.

In particular, if $m = 1$, the required function is nothing but

$$f^*(z) = (x_0 + x_1 z + \cdots + x_n z^n)^2 \equiv \sum_{\nu=0}^{n} C_\nu z^\nu, \; (\text{mod. } z^{n+1}).$$

For example, if $x_0+x_1z+\cdots+x_nz^n$ does not vanish in $|z|<1$, then, on account of the relation (3), CARATHÉODORY-FEJÉR function becomes

$$\varphi^*(z)=\varepsilon\,\frac{\bar{x}_n+\bar{x}_{n-1}z+\cdots+\bar{x}_0z}{x_0+x_1z+\cdots+x_nz^n}\equiv\sum_{\nu=0}^{n}a_\nu z^\nu,\ (\text{mod. } z^{n+1}),$$

and hence the required function must be given by

$$f^*(z)=(x_0+x_1z+\cdots+x_nz^n)^2.$$

An interesting question, which is here left open, is whether m is generally equal to 1 or not; if the former be the case, our problem might be completely solved.