103 On a Problem Proposed by Hardy and Littlewood.

(The Fourth Report on the Order of Linear Form.)

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1. We consider the function $\varphi_{\alpha,\beta}(t)$, which is the minimum absolute value of $t(\alpha x-y-\beta)$ for the integral values of x and y, where $|x|<t$. In the former reports I treated mainly the problem of finding the inferior limit of this function, which may be considered as an extension of a problem solved by Minkowski. On the other hand, Hardy and Littlewood have proposed in the paper "On some Problem of diophantine Approximation"[1] to determine the superior limit of this function, and Khintchine has proved that, if

$$\limsup \varphi_{\alpha,\beta}(t)=\infty \tag{1}$$

then the denominators (a_n) of the simple continued fraction for α can not be limited, and conversely, if they are not limited, then we can choose β, such that (1) subsists.[2] I wish to apply the same idea as in my former reports to this problem.

2. Let us consider on the xy-plane a system of lattice points, corresponding to the integral values of x and y, and the line L: $\alpha x-y-\beta=0$ and Y: $x=0$, whose intersection is supposed to be M. First we construct a parallelogramm, whose sides are parallel to L and Y and whose center is M and which contains no lattice point in it. We translate the upper and the lower sides (which are parallel to L) away from L till a lattice point $P_{n(k)}$ comes on one of these sides and again translate the left and the right sides (which are parallel to Y) away from Y till a lattice point $P_{n(k+1)}$ comes on one of these sides. Next we draw a parallel line to L through $P_{n(k+1)}$ and taking this line as the upper or lower side we construct the parallelogramm in a similar manner as above, which contains no lattice point in it, but the lattice point $P_{n(k+2)}$ on one of the right or left side, and so on. Thus we have a series of parallelogramms

$$S_{n(k)},\ S_{n(k+1)},\ S_{n(k+2)},\ \dots,$$

and of the points

$$P_{n(k)},\ P_{n(k+1)},\ P_{n(k+2)},\ \dots.$$

1) Acta Mathematica **37** (1914), pp. 155–191.

2) Über die angenäherte Auflösung linearer Gleichungen in ganzen Zahlen, Recueil de Mathématiques de Moscou, **32** (1924), pp. 203–219.

We can suppose $0<\alpha<1$, $0<\beta\leqq 1/2$. In this case we take as the first element $P_{n(1)}$: (0,0) and $S_{n(1)}$ the parallelogramm with the sides $x=\pm 1$, $y=\alpha x-\beta\pm\beta$. We call the points the principal approximate points and these parallelogramms the approximate parallelogramms. We see that, these principal approximate points and only these give us the best approximation of $\alpha x-y-\beta=0$ and that lim sup $\varphi_{\alpha,\beta}(t)=$ lim sup $I_{n(k)}$, where $4I_{n(k)}$ denotes the area of $S_{n(k)}$. If however, we, consider only the points for which $\alpha x-y-\beta>0$ or $\alpha x-y-\beta<0$, then we have other points to give the best approximation, which we call the intermediary approximate points. We arrange the principal and the intermediary approximate points in the order of $|x|$ and call them $P_1, P_2, P_3, \ldots.$ Then, to P_i corresponds the parallelogramm S_i whose area is $4I_i$, so that

$$S_n\equiv S_{n(k)} \text{ for } n=n(k)$$

and

$$I_n<I_{n(k+1)} \text{ for } n(k)<n<n(k+1).$$

Thus we have

$$\limsup \varphi_{\alpha,\beta}(t)=\limsup I_n .$$

3. Let $n(k)=i<n(k+1)$, then $P_{i+1}, P_{i+2}, \ldots, P_{n(k+1)-1}$ lies on the side opposite to P_i with respect to L. Let P_j be the last point, which lies on this side and for which $j<i$, and let P_m be one of the points $P_j, P_{i+1}, P_{i+2}, \ldots, P_{n(k+1)-1}$ and n the least number greater than i and m. As in the former reports we transform (by an affine transformation) P_i, P_m, P_n in (0,0) (0,−1), (1,0), (in particular, if P_n is the $P_{n(k+1)}$ and lies in the same side of L with P_i, then we transform these points in (−1,0), (0,0), (1,0)), and let the new position of L and Y be $L_n: \alpha_n x-y-\beta_n=0$ and $Y_n: \alpha_n' x+y+\beta_n'=0$. By the similar method as in the former reports we can find the sequence (q_n) and (μ_n) where $\mu_n=\pm 1$, which satisfy the following relations:

$$\left.\begin{aligned}
\alpha&=\frac{1}{q_1-}\,\frac{\mu_1}{q_2-}\,\frac{\mu_2}{q_3-}\cdots\cdots,\\
\alpha_n&=\frac{1}{q_{n+1}-}\,\frac{\mu_{n+1}}{q_{n+2}-}\,\frac{\mu_{n+2}}{q_{n+3}-}\cdots\cdots,\\
\beta&=\nu_1\alpha-\mu_1\nu_2\alpha\alpha_1+\mu_1\mu_2\nu_3\alpha\alpha_1\alpha_2-\cdots\cdots,\\
\beta_n&=\nu_{n+1}\alpha_n-\mu_{n+1}\nu_{n+2}\alpha_n\alpha_{n+1}+\mu_{n+1}\mu_{n+2}\nu_{n+3}\alpha_n\alpha_{n+1}\alpha_{n+2}-\cdots\cdots,\\
\text{where}\quad \nu_n&=\frac{\mu_n+1}{2},
\end{aligned}\right\}(3),$$

$$\left.\begin{aligned} \alpha_n' &= -\mu_n\left(a_n - \frac{\mu_{n-1}}{a_{n-1}-}\cdots\frac{\mu_1}{-a_1}\right), \\ \beta_n' &= \mu_n\nu_n + \frac{\mu_n\mu_{n-1}\nu_{n-1}}{\alpha'_{n-1}} + \frac{\mu_n\mu_{n-1}\mu_{n-2}\nu_{n-2}}{\alpha'_{n-1}\alpha'_{n-2}} + \cdots\cdots \end{aligned}\right|$$

and

$$\left.\begin{aligned} I_n &= \left|\frac{\beta_n(\alpha_n'+\beta_n')}{\alpha_n+\alpha_n'}\right| \text{ if } \beta_n \leqq 1/2 \\ &= \left|\frac{(1-\beta_n)(\alpha_n'+\beta_n')}{\alpha_n+\alpha_n'}\right| \text{ if } \beta > 1/2 \end{aligned}\right\} (4).$$

For the α_n' and β_n' we have the inequalities:

$$\left.\begin{aligned} &\text{if } \alpha_n' > 0 \quad \text{then } 1-\alpha_n' < 2\beta_n' < 1, \\ &\text{if } \alpha_n' < 0 \quad \text{then } \alpha_n' < -1 \text{ and } -\alpha_n' > 2\beta_n' > 1. \end{aligned}\right\} (5).$$

4. The continued fraction for α in (3), in which all a_n are smaller than 1, will be called half simple. Let p_n/q_n be the n-th. convergent of this continued fraction and $\left|\alpha - \frac{p_n}{q_n}\right| = \frac{1}{\lambda_n q_n^2}$, then we have

$$\lambda_n = |\,\alpha_n + \alpha_n'\,|. \qquad (6).$$

We see that p_n/q_n is a principal or intermediary convergent of the simple continued fraction for α and that I_n can not be very large unless $\alpha_n + \alpha_n'$ be not very small, because of (5). λ_n can be very small, when and only when p_n/q_n is an intermediary convergent, whose representative in Klein's interpretation lies in the middle of a long side of the approximate polygon. Therefore we see that I_n can not be very large unless (a_n) be not limited. We can also prove the convere theorem by the same method.

5. This method shows us a precise relation between (a_n) and lim $\varphi_{\alpha,\beta}(t)$, which can apply to many other problems. For example we can prove the following theorem:

"*The necessary and sufficient condition, that β can be so chosen that* lim sup $\varphi_{\alpha,\beta}(t)=0$, *is that (a_n) are not limited.*"