98. On the System of Linear Inequalities and Linear Integral Inequality.

By Matsusaburô FUJIWARA, M. I. A.
Mathematical Institute, Tohoku Imperial University, Sendai.
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1. Dines and Carver[1] have treated in several papers the condition for the inconsistency of a system of linear inequalities

$$L_i(u)=a_{i1}u_1+a_{i2}u_2+\ldots\ldots+a_{in}u_n\geqq 0,\ (i=1,2,\ldots\ldots,m) \quad (1)$$

and Dines[2] has extended their investigation to the problem of linear integral inequality. On the other hand Kakeya[3] has treated, as an application of his theory on the system of linear integral equations, the problem of linear differential inequality. It is, however, not remarked by any one that these three problems belong to the same category.

2. Taking this fact into account we can solve the problem of linear inequality in the following manner.

First consider the system of non-homogeneous linear equations

$$L_i(u)=b_i,\ (i=1,2,\ldots\ldots,m) \quad (2)$$

and its adjoint system

$$M_k(v)=a_{1k}v_1+a_{2k}v_2+\ldots\ldots+a_{mk}v_m=0. \quad (3)$$

Then there exists the relation

$$\sum_i v_iL_i(u)-\sum_k u_kM_k(v)=0. \quad (4)$$

Let $v_1^{(\lambda)},\ v_2^{(\lambda)},\ \ldots\ldots,\ v_m^{(\lambda)},\ (\lambda=1,2,\ldots\ldots,s)$

be linearly independent solutions of (3). ($s=m-r$ if r be the rank of the matrix (a_{ik})).

It will easily be proved that (2) has solution when and only when $(b_1, b_2,\ldots\ldots, b_m)$ satisfy the condition

$$\sum_i b_i v_i^{(\lambda)}=0,\ \lambda=1,2,\ldots\ldots,s. \quad (5)$$

1) Dines, Annals of Math., (2) **20** (1918–19), **27** (1925–26), **28** (1928), p. 41, 386. Carver, ibid. (2) **23** (1923).
2) Dines, Trans. American Math. Society, **30** (1928), Annals of Math., (2) **28** 1926–27), p. 393.
3) Kakeya, Proc. Math.-Phy. Soc. Japan. (2) **8** (1915). See also Fujiwara, Science Reports, Tohoku University, **4** (1915).

For the existence of the solution of (1) it is necessary and sufficient that (5) has the non-negative solutions $b_1, b_2, \ldots, b_m \geqq 0$, that is, the origin lies within or on the boundary of the least convex polyhedron K in the s-dimensional space, which contains m points $P_1, P_2, \ldots, P_m$, whose Cartesian coordinates are respectively

$$(v_i^{(1)}, v_i^{(2)}, \ldots, v_i^{(s)}).$$

In order that $\quad L_i(u) > 0, \ (i=1, 2, \ldots, m) \qquad (1')$

it is necessary and sufficient that the origin lies within K.

This can be seen very easily by means of the idea of centre of mass.

The condition given by Dines and Carver for the inconsistency of (1) (or (1′)) is that (3) has positive (or non-negative) solutions $v_1, v_2, \ldots, v_m > 0$ (or $\geqq 0$).[4]

This is of course equivalent with the condition above stated, which can be directly verified without difficulty.

The latter condition may also be stated as follows.

For the inconsistency of (1) (or (1′)) it is necessay and sufficient that the origin lies within or on the boundary (or within) of the least convex polyhedron K' which contains m points $Q_1, Q_2, \ldots, Q_m$ whose Cartesian coordinates are respectively

$$(a_{i1}, a_{i2}, \ldots, a_{in}).$$

K' is r-dimensional, when the rank of the matrix (a_{ik}) is equal to r.

That the completely signed matrix, introduced by Dines, satisfies this condition is self-evident.

3. Similarly we can treat the linear integral inequality

$$L(\varphi) = \varphi(x) - \lambda \int_a^b K(x, y)\, \varphi(y)\, dy \geqq 0. \qquad (6)$$

Let
$$M(\psi) = \psi(x) - \lambda \int_a^b K(y, x)\, \psi(y)\, dy = 0 \qquad (7)$$

be the adjoint integral equation to $L(\varphi)=0$, and $\psi_1(x), \psi_2(x), \ldots, \psi_m(x)$ be the linearly independent solutions of (7). Then there exists the identity

$$\int_a^b [\psi L(\varphi) - \varphi M(\psi)] dx = 0. \qquad (8)$$

If λ is no characteristic value, then there is no solution of (7) except $\psi(x)=0$, while

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4) Stiemke has treated the inverse problem in Math. Annalen **76** (1915), and showed that, when (1) has solution, then (3) has the positive solution.

$$L(\varphi)=f(x) \tag{9}$$

has solution, whatever $f(x)$ may be, consequently (6) has solution. If λ be a characteristic value, then (9) has solution when and only when $f(x)$ satisfies

$$\int_a^b f(x)\psi_i(x)dx=0, \ (i=1, 2,\ldots, m), \tag{10}$$

as was shown by Schmidt. Whence follows the following theorem.

For the existence of the solution of (6) it is necessary and sufficient that (10) has the solution $f(x)\geqq 0$, that is, the origin lies within the least convex body K containing the curve in the m-dimensional space, represented parametrically by

$$x_1=\psi_1(t),\ x_2=\psi_2(t),\ldots,\ x_m=\psi_m(t),\ (a\leqq t\leqq b),$$

if $\psi_1(x)$, $\psi_2(x),\ldots,\psi_m(x)$ are not linearly dependent in any subinterval of (a, b). If $\psi_i(x)$ are linearly dependent in some subinterval, then we have to add some portion of the boundary of K as the domain in which the origin should lie.

This is the direct consequence of Kakeya's theorem.[5]

Dines[2] has stated the required condition in such a form, that any linear combination of $\psi_1(x)$, $\psi_2(x),\ldots,\psi_m(x)$ should change its sign on the interval (a, b). This follows immediately from the condition above given, and that the completely signed set of functions satisfies the condition is also clear from our form of condition.

4. We will recapitulate here the problem of linear differential inequality to show the similarity with the problems of linear inequality and linear integral inequality above discussed.

Consider $$L_i(u)=\frac{du_i}{dx}+a_{i1}(x)u_1+a_{i2}(x)u_2+\ldots+a_{in}(x)u_n\geqq 0,$$
$$(i=1, 2,\ldots, n), \tag{11}$$

and $$M_k(v)=-\frac{dv_k}{dx}+a_{1k}(x)v_1+a_{2k}(x)v_2+\ldots+a_{nk}(x)v_n=0,$$
$$(k=1, 2,\ldots, n). \tag{12}$$

Between them there exists an identity

$$\int_a^b\sum\Big(v_iL_i(u)-u_iM_i(v)\Big)dx=\Big[\sum u_iv_i\Big]_a^b, \tag{13}$$

which corresponds to (4) and (8).

5) Kakeya, Tohoku Math. Journ. **4**, (1913–14).

Let $$(v_1^{(\lambda)}, v_2^{(\lambda)}, \ldots, v_n^{(\lambda)}) \quad (\lambda=1, 2, \ldots, n)$$

be the fundamental system of (12). Then the solution of

$$L_i(u)=f_i(x), \quad (i=1, 2, \ldots, n) \tag{14}$$

exists under the condition that the values of (u_i) at $x=a$, $x=b$ are assigned, when and only when $(f_i(x))$ satisfy the condition

$$\int_a^b \sum f_i(x) v_i^{(\lambda)}(x) dx = \Big[\sum u_i v_i^{(\lambda)}\Big]_a^b = c_\lambda \text{ (say)}. \tag{15}$$

For the existence of the solutions of (11) under the given boundary condition it is necessary and sufficient that (15) has the solutions $f_i(x) \geqq 0$. This condition can be modified as follows.

Let K_λ be the least convex body containing the curve in the n dimensional space

$$x_1=v_1^{(\lambda)}(t), \; x_2=v_2^{(\lambda)}(t), \ldots, \; x_n=v_n^{(\lambda)}(t)$$

and let K be the mean body of $K_1, K_2, \ldots, K_n$. Further let D be the envelopping cone of K, having the origin as the vertex. (D will coincide with the whole space, when K contains the origin). Then the required condition is that the point $(c_1, c_2, \ldots, c_n)$ lies within D.

The case of the linear differential inequality of the form

$$L(u)=u^{(n)}+p_1(x)u^{(n-1)}+\ldots+p_n(x)u \geqq 0, \tag{16}$$

when the values of $u, u', u'', \ldots u^{(n-1)}$ are assigned at $x=a, b$, can also be treated similarly, by introducing the adjoint differential expression $M(v)$ to $L(u)$, and making use of the well known relation

$$\int_a^b \Big(vL(u)-uM(v)\Big)dx = \Big[P(u, v)\Big]_a^b. \tag{17}$$

This result is essentially due to Kakeya.