PAPERS COMMUNICATED

172 *Differential Geometry of Conics in the Projective Space of Three Dimensions.*

IV. Remarks and corrections on the previous papers.

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I have found out some incomplete points on the statements of the fundamental theorems in my previous papers.[1] I will, therefore, now withdraw both of the fundamental theorems, but all other parts in these papers which are independent of them are correct. As a substitute for them I will state again in this paper the fundamental theorems in correct forms, by proceeding in a different direction, which enable us to discuss the theory by a very simple method. The imperfections of the fundamental theorems lie upon the fact that under the conditions stated there the family of conics is not uniquely determined except for projective transformations, i.e. any two families which have the same invariants stated there may not be transformed to each other by a projective transformation, but it is uniquely determined except for such transformations, that they transform any planes in the same manner as projective transformations do and also points on every plane to points on the same plane projectively.

1. *Four kinds of families of conics.* There are following four kinds of families of conics: possessing the properties (1) all the planes $\mathfrak{l}$ have no common points, (2) all the planes $\mathfrak{l}$ pass through a fixed point, (3) all the planes $\mathfrak{l}$ have a common straight line, (4) all the planes $\mathfrak{l}$ coincide with each other, i.e. the coordinates $\mathfrak{l}$ are constants. The fourth kind is not different from the families of conics on the plane, on which the fundamental theorem was stated and proved in another papers of mine[2]. The second and the third kinds will be discussed in the next paper. In this paper I will consider only the first kind, which is the most general case.

2. *One-parameter family.* Let $\mathfrak{l}$ be the projectively normalized coordinates in Sannia's sense[3] and σ the projective length, and consider

1) Differential geometry of conics in the projective space of three dimensions, I and III, Proc., **4** (1928), 258 and 344.

2) Über projective Differentialgeometrie I, Tôhoku Math. Journ. **28** (1927), 126-148 and Differential geometry of conics in the projective space of three dimensions II, Proc., **4** (1928), 337-340.

3) See G. Sannia, Nuova trattazione della geometria proiettivo-differenziale delle curve sghembe I, II, Annali di Matematica, ser. 4, **1**, **3** (1923-25).

the developable surface enveloped by $\mathfrak{l}(\sigma)$, then we have three invariant lines on the every plane $\mathfrak{l}$, which are respectively intersectinons of planes $\mathfrak{l}'$, $\mathfrak{l}''$, $\mathfrak{l}'''$ with $\mathfrak{l}$. The coordinates of these lines on the plane $\mathfrak{l}$ can be expressed by

$$\begin{aligned}
\lambda_1^{(i)} &= \rho(l_1^{(i)} l_2 l_3 l_4 - l_1 l_2^{(i)} l_3 l_4 - l_1 l_2 l_3^{(i)} l_4 + l_1 l_2 l_3 l_4^{(i)}) , \\
\lambda_2^{(i} &= \rho(l_1^{(i)} l_2 l_3 l_4 - l_1 l_2^{(i)} l_3 l_4 + l_1 l_2 l_3^{(i)} l_4 - l_1 l_2 l_3 l_4^{(i)}) , \\
\lambda_3^{(i)} &= \rho(l_1^{(i)} l_2 l_3 l_4 + l_1 l_2^{(i)} l_3 l_4 - l_1 l_2 l_3^{(i)} l_4 - l_1 l_2 l_3 l_4^{(i)}) ,
\end{aligned}$$

where $\rho = (\Pi l_i)^{-\frac{3}{8}}$ and $i=1, 2, 3$.

Now we can find very easily the six invariants

$$I^{ij} = A^{\alpha\beta} \lambda_\alpha^{(i)} \lambda_\beta^{(j)} \qquad i, j = 1, 2, 3,$$

among which there is a relation

$$|I^{ij}| = \text{const.},$$

because by the rule in the theory of determinants

$$\begin{aligned}
|I^{ij}| &= |A^{\alpha\beta}| \, |\lambda_\gamma^{(i)}|^2 = |l_\gamma^{(i)}|^2 \\
&= \text{const.} \begin{vmatrix} l_1 & l_1' & l_1'' & l_1''' \\ l_2 & l_2' & l_2'' & l_2''' \\ l_3 & l_3' & l_3'' & l_3''' \\ l_4 & l_4' & l_4'' & l_4''' \end{vmatrix} = \text{const.},
\end{aligned}$$

where $A^{\alpha\beta}$ denote the dual coordinates of conics $\mathfrak{a}(\sigma)$. Hence five of these invariants are essential and we can prove the fundamental theorem, if we denote the dual projective curvature and torsion[1] by r and t:

When seven invariants r, t, I^{ij} are given as the functions of σ, then the one-parameter family of conics in the projective space of three dimensions is uniquely determined, except for projective transformations.

3. *Two-parameter family.* Let $\mathfrak{l}$ be the projectively normalized coordinates in Fubini's sense[2] with regard to the surface enveloped by $\mathfrak{l}(u^1, u^2)$, then we can introduce three lines on every plane $\mathfrak{l}$, of which two lines are the intersections of planes $\mathfrak{l}_i$ $(i=1, 2)$ with $\mathfrak{l}$ and another one is the second normal[3]. The coordinates of these lines on the plane $\mathfrak{l}$ can be represented in the same manner as in No. 2 by

$$\begin{aligned}
\lambda_{1,i} &= \rho(l_{1,i} l_2 l_3 l_4 - l_1 l_{2,i} l_3 l_4 - l_1 l_2 l_{3,i} l_4 + l_1 l_2 l_3 l_{4,i}) , \\
\lambda_{2,i} &= \rho(l_{1,i} l_2 l_3 l_4 - l_1 l_{2,i} l_3 l_4 + l_1 l_2 l_{3,i} l_4 - l_1 l_2 l_3 l_{4,i}) , \\
\lambda_{3,i} &= \rho(l_{1,i} l_2 l_3 l_4 + l_1 l_{2,i} l_3 l_4 - l_1 l_2 l_{3,i} l_4 - l_1 l_2 l_3 l_{4,i}) ,
\end{aligned}$$

1) G. Sannia, loc. cit.
2) See Fubini-Čech, Geometria proiettiva differenziale I, 1926, 85-87.
3) See Fubini-Čech, loc. cit., 87.

$$\mu_1 = \rho(p_1 l_2 l_3 l_4 - l_1 p_2 l_3 l_4 - l_1 l_2 p_3 l_4 + l_1 l_2 l_3 p_4),$$
$$\mu_2 = \rho(p_1 l_2 l_3 l_4 - l_1 p_2 l_3 l_4 + l_1 l_2 p_3 l_4 - l_1 l_2 l_3 p_4),$$
$$\mu_3 = \rho(p_1 l_2 l_3 l_4 + l_1 p_2 l_3 l_4 - l_1 l_2 p_3 l_4 - l_1 l_2 l_3 p_4),$$

where

$$\mathfrak{p} = \mathfrak{l}_{ij} g^{ij},$$

g^{ij} being the Fubini's fundamental quantities[1]. Then we can get two covariant differential forms

$$S_{ij}\, du^i\, du^j = A^{\alpha\beta} \lambda_{\alpha,i} \lambda_{\beta,j}\, du^i\, du^j,$$
$$T_i\, du^i = A^{\alpha\beta} \mu_\alpha \lambda_{\beta,i}\, du^i$$

and an invariant

$$U = A^{\alpha\beta} \mu_\alpha \mu_\beta,$$

among which there exists a relation

$$\begin{vmatrix} S_{11} & S_{12} & T_1 \\ S_{21} & S_{22} & T_2 \\ T_1 & T_2 & U \end{vmatrix} = \text{const.}$$

similarly as in No. 2. It follows therefore that U must be expressed by S_{ij} and T_i.

Now we can prove the fundamental theorem :

When the Fubini's three differential forms $g_{ij}du^i du^j$, $a_{ijk}du^i du^j du^k$, $q_{ij}du^i du^j$[2] *and two other forms $S_{ij}du^i du^j$, $T_i du^i$ are given, the two-parameter family of conics having these forms in the projective space of three dimensions is uniquely determined, except for projective transformations.*

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1) Fubini-Cech, loc. cit., 64-67, where this form is expressed by the notation $a_{rs}\, du^r\, du^s$.

2) Fubini-Čech, loc. cit., 68-81.