PAPERS COMMUNICATED

1. On a System of Generalized Orthogonal Functions.

By Matsusaburô FUJIWARA, M.I.A.
Mathematical Institute, Tohoku Imperial University, Sendai.

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Let $J_0(x)$ be the Bessel's function of order zero, and consider the system of functions consisting of

$$\varphi(x, \lambda)=\sqrt{\pi \lambda x}\; J_0(\lambda x), \quad (\lambda \geqq 0,\; x \geqq 0).$$

This forms a system of generalized orthogonal functions in the sense that I have defined in the previous paper in these Proceedings, 2 (1926), that is,

$$M\left\{\varphi(x, \lambda)\varphi(x, \mu)\right\}=\delta_{\lambda\mu}, \quad (\delta_{\lambda\mu}=1 \text{ for } \lambda=\mu, \; =0 \text{ for } \lambda \neq \mu)$$

where $M\left\{f(x)\right\}$ denotes $\lim \frac{1}{T}\int_0^T f(x)\,dx$ for $T \rightarrow \infty$.

This fact follows immediately from the asymptotic formula

$$\sqrt{\pi x}\; J_0(x)=\sqrt{2}\; \cos\left(\frac{\pi}{4}-x\right)+O\left(\frac{1}{x}\right)$$

and the following

Lemma. If $f(x)$, $g(x)$ are integrable in every finite interval in $(0, \infty)$, and $M\left\{f(x)\right\}$ exists, then $M\left\{g(x)\right\}=M\left\{f(x)\right\}$, provided that $g(x)=f(x)+O\left(\frac{1}{x}\right)$.

As a generalization of the almost periodic functions of Bohr, I have treated in the previous paper a class of functions (F), uniformly approximable by a linear combination of functions in the system $\left\{\varphi(x, \lambda)\right\}$ for $x \geqq 0$, and have deduced Parseval's formula for the class (F).

Recently Prof. Wiener has deduced in his important paper, the spectrum of an arbitrary function, Proc. London Math. Society, 27 (1928), the Parseval's formula for the almost periodic functions very ingeniously by applying his theory of the spectrum of arbitrary functions. I wish here to remark that Wiener's theory can also be applicable to the class (F).

Let $f(x)$ be a function satisfying the following conditions:

(A) $\mathfrak{M}\left\{|f(x)^2|\right\}$ exists, where $\mathfrak{M}\left\{h\right\}$ denotes $\lim \frac{1}{2T}\int_{-T}^{T} h(x)\,dx$ for $T\to\infty$.

(B) $g(t)=\mathfrak{M}\left\{f(x)\bar{f}(x+t)\right\}$ exists uniformly in t $(-\infty<t<\infty)$, where $\bar{f}$ denotes the conjugate complex function of f.

Then after Wiener we can prove:

(1) $R(u)=\frac{1}{\pi}\int_{-\infty}^{\infty} g(t)\frac{\sin ut}{t}\,dt$ exists and is non-decreasing and of limited variation.

(2) $R(u)=\frac{1}{2}\left\{R(u+0)+R(u-0)\right\}$.

Therefore $R(u)$ can have only discontinuous points of the first kind. Let $\lambda_1, \lambda_2, \ldots\ldots$ be all discontinuous points on the interval $(0, \infty)$, and put

$$R(+0)=a_0, \quad R(\lambda_k+0)-R(\lambda_k-0)=a_k.$$

Then

(3) $a_k>0$, and $R(\infty)$ exists and $\leqq g(0)=\mathfrak{M}\left\{|f(x)|^2\right\}$.

If in addition to (A), (B), the condition:

(C) $g(t)$ is continuous at $t=0$,

be assumed, then we have further

(4) $R(\infty)=g(0)$; and $\sum a_k$ is convergent, $\leqq R(\infty)$.

(5) If $\gamma(t)=\left(\text{Real part of } g(t)\right)-\sum a_k\cos\lambda_k t$, then $\mathfrak{M}\left\{|\gamma(t)|^2\right\}=0$.

If $f(x)$ be assumed to be almost periodic, then the conditions (A), (B), (C) are satisfied, and $g(t)$, consequently $\gamma(t)$, is also almost periodic. Whence follows $\gamma(t)=0$, that is:

(6) Real part of $g(t)=\sum a_k\cos\lambda_k t$.

(7) $\sum a_k=$ Real part of $g(0)=\mathfrak{M}\left\{|f(x)|^2\right\}$.

(8) $R(u)=\sum_{\lambda_k<u} a_k+\begin{cases} a_j/2 & \text{if } u=\lambda_j,\\ 0 & \text{if } u\neq\lambda_j.\end{cases}$

From this formula Prof. Wiener has deduced the Parseval's formula.

Now returning to our case, let $F(x)$ be a real function belonging to the class (F). Then it can easily be proved that

$$G(t) = M\left\{F(x)\,F(x+t)\right\}, \quad \text{for } t \geqq 0$$

exists uniformly and is uniformly approximable by a linear combination of functions $\left\{\cos \lambda t\right\}$.

If we put

$$f(x) = F(x) \quad \text{for} \quad x \geqq 0, \quad = iF(-x) \quad \text{for} \quad x < 0,$$

and

$$g(t) = \mathfrak{M}\left\{f(x)\bar{f}(x+t)\right\},$$

then

$$M\left\{|F(x)|^2\right\} = \mathfrak{M}\left\{|f(x)|^2\right\},$$

$$g(t) = G(t) \quad \text{for} \quad t \geqq 0, \quad g(t) = G(-t) \quad \text{for} \quad t < 0;$$

for $f(x)$, $g(t)$ the conditions (A), (B), (C) hold good, and $g(t)$ is almost periodic. Therefore all the theorems (1)—(8) are valid.

Since in our case $g(t)$ is real, we have from (6)

$$a_k/2 = M\left\{g(t)\cos\lambda_k t\right\} = M\left\{G(t)\cos\lambda_k t\right\}$$

$$= \lim_{T\to\infty}\frac{1}{T}\int_0^T \cos\lambda_k t\,dt\left[\lim_{S\to\infty}\frac{1}{S}\int_0^S F(x)\,F(x+t)\,dx\right]$$

$$= \lim\frac{1}{S}\int_0^S F(x)\,dx\left[\lim\frac{1}{T}\int_0^T F(x+t)\cos\lambda_k t\,dt\right]$$

$$= \lim\frac{1}{S}\int_0^S F(x)\,dx\left[\lim\frac{1}{T}\int_x^{x+T} F(u)\cos\lambda_k(u-x)\,du\right],$$

(9) $$a_k/_2 = M\left\{F(x)\cos\lambda_k x\right\}^2 + M\left\{F(x)\sin\lambda_k x\right\}^2,$$

for the order of limits is interchangeable by the uniform convergence of $\frac{1}{S}\int_0^S F(x)\,F(x+t)\,dx$.

Since

$$\psi(x,\lambda) = \sqrt{\pi\lambda x}\;Y_0(\lambda x) = \sqrt{2}\sin\left(\frac{\pi}{4} - \lambda x\right) + O\left(\frac{1}{x}\right),$$

where $Y_0(x)$ denotes the Bessel's function of the second kind, we have by the lemma

$$M\left\{\varphi(x, \lambda)\psi(x, \mu)\right\} = 0, \quad M\left\{F(x)\psi(x, \lambda)\right\} = 0,$$

$$M\left\{F(x)\varphi(x, \lambda)\right\} = M\left\{F(x)\cos\lambda x\right\} + M\left\{F(x)\sin\lambda x\right\},$$

and $$M\left\{F(x)\psi(x, \lambda)\right\} = M\left\{F(x)\cos\lambda x\right\} - M\left\{F(x)\sin\lambda x\right\}.$$

From (7), (9) we have finally the Parseval's formula

$$\sum a_k = \sum M\left\{F(x)\varphi(x, \lambda_k)\right\}^2$$

$$= g(0) = M\left\{|F(x)|^2\right\}.$$