PAPERS COMMUNICATED

90. On the System of Linear Inequalities.

By Matsusaburô FUJIWARA, M.I.A.
Mathematical Institute, Tohoku Imperial University, Sendai.
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In my former paper, On the system of linear inequalities and linear integral inequality, this Proceedings, 4 (1928), I have shown that the condition for the existence of the solutions of a system of linear inequalities is equivalent to the condition for the existence of *positive* solutions of a certain system of linear equations, and have given the said necessary and sufficient condition in the following form.

In order that the system of linear equations

$$\sum_{k=1}^{m} a_{ik}x_k=0\,, \qquad (i=1,\ 2,\ \ldots\ldots,\ n)$$

has a system of positive solutions $x_1, x_2, \ldots\ldots, x_n > 0$, it is necessary and sufficient that the origin must lie within the smallest convex polyhedron S containing the following m points in the n-dimensional space:

$$P_i(a_{1i},\ a_{2i},\ \ldots\ldots,\ a_{ni})\,, \qquad (i=1,\ 2,\ \ldots\ldots,\ m).$$

Recently, in a paper, Huber: Eine Erweiterung der Dirichletschen Methode des Diskontinuitätsfaktors und ihre Anwendung auf eine Aufgabe des Wahrscheinlichkeitsrechnung, Monatshefte für Mathematik und Physik 37 (1930), the following theorem and its proof due to Furtwängler are published:

If at least one of r linear forms

$$L_i(x)=\sum_{k=1}^{n} a_{ik}x_k\,, \qquad (i=1,\ 2,\ \ldots\ldots,\ r) \tag{1}$$

for any system of non-negative $x_1, x_2, \ldots\ldots, x_n \geqq 0$ becomes positive, then there exists a system of positive numbers $p_1, p_2, \ldots\ldots, p_r > 0$ such that

$$M_i(p)=\sum_{k=1}^{r} a_{ki}p_k > 0\,, \qquad (i=1,\ 2,\ \ldots\ldots,\ n). \tag{2}$$

The existence of a system of positive numbers $p_1, p_2, \ldots\ldots, p_r$ such that the relation (2) holds good is nothing but the existence of the *positive* solutions $x_1, x_2, \ldots\ldots, x_{n+r}$ for the system of linear equations:

$$\sum_{k=1}^{r} a_{ki}x_k=x_{r+i}\,, \qquad (i=1,\ 2,\ \ldots\ldots,\ n).$$

It is therefore necessary and sufficient for the validity of (2), that the origin lies within the smallest convex polyhedron S containing the following $n+r$ points in the n-dimensional space:

$$\begin{array}{ll} P_1(a_{11}, a_{12}, \ldots\ldots, a_{1n}), & Q_1(-1, 0, 0, \ldots\ldots, 0), \\ P_2(a_{21}, a_{22}, \ldots\ldots, a_{2n}), & Q_2(0, -1, 0, \ldots\ldots, 0), \\ \cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots \\ P_r(a_{r1}, a_{r2}, \ldots\ldots, a_{rn}), & Q_n(0, 0, 0, \ldots\ldots, -1). \end{array}$$

It is not difficult to show that this condition is equivalent to the Furtwängler's condition.

Suppose that S does not contain the origin within it. Then the points $P_1, P_2, \ldots\ldots, P_r$ must lie on one and the same side of a suitable hyperplane

$$\pi: \quad A_1x_1+A_2x_2+\cdots\cdots+A_nx_n=0\,,$$

passing through the origin, with the points $Q_1, Q_2, \ldots\ldots, Q_n$ (some one may lie on this plane). In order that Q_1 and Q_2 lie on the same side of π, it is necessary that $-A_1$ and $-A_2$ have the same sign. It follows then that $A_1, A_2, \ldots\ldots, A_n \geqq 0$ (or $\leqq 0$). Assume that $A_1, A_2, \ldots\ldots, A_n \geqq 0$. Then from the fact that $P_1, P_2, \ldots\ldots, P_r$ lie on the same side of π with $Q_1, Q_2, \ldots\ldots, Q_n$, we must have

$$\begin{aligned} L_1(A)&=a_{11}A_1+a_{12}A_2+\cdots\cdots+a_{1n}A_n \leqq 0\,, \\ L_2(A)&=a_{21}A_1+a_{22}A_2+\cdots\cdots+a_{2n}A_n \leqq 0\,, \qquad (3) \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ L_r(A)&=a_{r1}A_1+a_{r2}A_2+\cdots\cdots+a_{rn}A_n \leqq 0\,. \end{aligned}$$

If the Furtwängler's condition is satisfied, that is, if at least one of linear forms (1) for any system of non-negative values of the variables has a positive value, it contradicts the relations (3) above deduced. Therefore S must contain the origin within it.

Conversely, if S contains the origin within it, then for any system

$$A_1,\ A_2,\ \ldots\ldots,\ A_n \geqq 0$$

at least one of the points $P_1, P_2, \ldots\ldots, P_r$ must lie on the side of π opposite to $Q_1, Q_2, \ldots\ldots, Q_n$. Therefore, at least one of $L_1(A)$, $L_2(A)$, $\ldots\ldots$, $L_r(A)$ must be > 0.

Thus it is shown that *the Furtwängler's condition is, not only sufficient, but also necessary for the existence of positive numbers* $p_1, p_2, \ldots\ldots, p_n$ *such that*

$$M_i(p)=\sum_{k=1}^{r} a_{ki}p_k > 0 \qquad (i=1, 2, \ldots\ldots, n).$$