# 100. *The Foundation of the Theory of Displacements.*

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In a previous paper[1] I have stated the most general displacement in a generalized manifold of finite dimensions, which contains all kinds of displacements as its special ones. Considering a new kind of space[2], I shall set out here the foundation of the theory of displacements from the stand-point of the theory of abstract spaces and define an abstract displacement, from which not only all kinds of displacements in a manifold of finite dimensions, but also those in the Hilbertian or the function manifold, are deduced by specialization.

1. We shall take an underlying manifold $M$ and a series of finite or enumerably infinite number of manifolds[3] having the sense of "neighbourhood": $M^{(1)}, M^{(2)}, \ldots\ldots$, and denote any one of their elements by $a, a^{(1)}, a^{(2)}, \ldots\ldots$ respectively. We associate to every set of the elements $a^*=(a, a^{(1)}, a^{(2)}, \ldots\ldots)$, consisting of every single element of the above manifolds, one of the mutually isomorphic manifolds $\overline{M}$, which are the spaces $(K)$ and are in general independent of the manifolds $M$ and $M^{(i)}$. The totality of all the sets $a^*$ forms evidently a manifold $M^*$ having the sense of "neighbourhood."

2. The displacements $\overline{M}(a^*)\to\overline{M}(b^*)$ are characterized by the following three axioms[4] for the one-to-one and bicontinuous[5] correspondences, in which every element of $\overline{M}(a^*)$ associated to a set $a^*$ corresponds to one of that $\overline{M}(b^*)$ associated to a set $b^*$.

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1) A. Kawaguchi, Theory of connections in the generalized Finsler manifold, II, Proc. 8 (1932), 340–343.

2) I name this space *the space* $(K)$, which is equal to the general vector space by S. Banach, where the idea "neighbourhood" stands for that "metrics," that is, a linear space having the sense of "neighbourhood." This space is therefore isomorphic to an abstract continuous group. S. Banach, Théorie des opérations linéaires, Warszawa 1932.

3) The "manifold" means in this paper the space $(L)$ in Fréchet's sense. M. Fréchet, Les espaces abstraits, Paris, 1926.

4) These axioms were first introduced by Veblen and Whitehead into a manifold of finite dimensions. O. Veblen-J. H. Whitehead, The foundations of differential geometry, 1932.

5) Continuity of a correspondence means that by the correspondence every open set corresponds to another open set.

(1) For any two sets $a^*$ and $b^*$ there exists at least one of the diplacements $\overline{M}(a^*)\rightarrow\overline{M}(b^*)$.

(2) By a combination of two displacements $\overline{M}(a^*)\rightarrow\overline{M}(b^*)$ and $\overline{M}(b^*)\rightarrow\overline{M}(c^*)$ follows always a displacement $\overline{M}(a^*)\rightarrow\overline{M}(c^*)$.

(3) The inverse correspondence of a displacement $\overline{M}(a^*)\rightarrow\overline{M}(b^*)$ is also a displacement $\overline{M}(b^*)\rightarrow\overline{M}(a^*)$.

3. The totality of the displacements forms a pseudo-group. The number of the displacements from $\overline{M}(a^*)$ to $\overline{M}(b^*)$ is not necessarily one. Moreover there may exist many displacements which carry $\overline{M}(a^*)$ into itself. These displacements form a sub-group $G$ of the group of self-correspondences. $G$ is called the *holonomic group* at the set $a^*$ and we can show that the holonomic groups are simply isomorphic to one another. When the holonomic group contains only one element, we say the displacement is holonomic. The manifold $M$ with a holonomic displacement is called a *flat manifold*; otherwise it is called a *curved manifold*.[1)]

We can conclude from the axioms that between the set of the displacements from $\overline{M}(a^*)$ to $\overline{M}(b^*)$ and the holonomic group there exists a one-to-one correspondence. Therefore the manifold is flat, when there exists only one displacement from $\overline{M}(a^*)$ to $\overline{M}(b^*)$; otherwise it is curved.

4. Now from our standpoint the geometry can be interpretated as the theory of manifolds $M$, $M^{(i)}$, $\overline{M}$, and of the displacements, that is to say, the manifolds and the displacements are the foundation of the geometry and by these the geometries are characterized.

Let $\overline{M}$ be coincide with $M$ and take the identical self-correspondence as the displacement, then $M$ is a flat manifold. The space in Klein's sense[2)] will be deduced from this manifold by associating a group of self-correspondences. The space in Cartan-Schouten's sense[3)] is also clearly a special one of our general case.

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1) This definition for flatness can be generalized to an abstract set. Let $m$ be an abstract set and an arbitrary set $\overline{m}$ be associated to every element of $m$. The displacements are then defined from the one-to-one correspondences between the sets $\overline{m}$ associated to any two elements of $m$ by three axioms analogous to the above ones. If the holonomic group of displacements contains only one element, then the set $m$ is flat.

2) F. Klein, Erlanger Programm, Gesammelte Math. Abh. I, Berlin, 1921.

3) J. A. Schouten, Erlanger Programm und Übertragungslehre. Neue Gesichtspunkte zur Grundlegung der Geometrie, Rendiconti del Circolo Matematico di Palermo, **50** (1926), 142–169.

5. We represent a displacement from $\overline{M}(a^*)$ to $\overline{M}(b^*)$ by $D_{a^*b^*}$ and the correspondence between elements in this displacement by the following expression:

$$\bar{a}_{b^*}=\bar{a}_{b^*}(\bar{a}_{a^*}, D_{a^*b^*}), \tag{1}$$

which is a one-valued set-function, continuous with respect to $\bar{a}_{a^*}$ in the manifold $\overline{M}(a^*)$. From the axioms it follows

$$\bar{a}_{c^*}(\bar{a}_{b^*}(\bar{a}_{a^*}, D_{a^*b^*}), D_{b^*c^*})=\bar{a}_{c^*}(\bar{a}_{a^*}, D_{a^*c^*}), \tag{2}$$

when the displacement $D_{a^*c^*}$ follows from combination of the two displacements $D_{a^*b^*}$ and $D_{b^*c^*}$;

$$\bar{a}_{a^*}=\bar{a}_{a^*}(\bar{a}_{b^*}(\bar{a}_{a^*}, D_{a^*b^*}), D_{b^*a^*}), \tag{3}$$

when $D_{b^*a^*}$ is the inverse of the displacement $D_{a^*b^*}$. We say two elements $\bar{a}_{a^*}$ and $\bar{a}_{b^*}$ related in (1) are *equivalent* to each other.

6. With respect to an underlying isomorphism between any two manifolds $\overline{M}$, let us represent the corresponding elements by the same letters, which are called *equal elements*. The underlying isomorphism may be defined arbitrarily but uniquely. Let $\bar{a}_{a^*}(a^*)$ be an element in $\overline{M}_{a^*}$ determined uniquely for every $a^*$, and continuous with respect to $a^*$, when we consider instead of $\bar{a}_{a^*}(a^*)$ their equal elements in some manifold $\overline{M}$. The change in $\bar{a}_{a^*}(a^*)$ due to a change from $a^*$ to $b^*$,

$$\Delta\bar{a}_{a^*b^*}=\bar{a}_{b^*}-\bar{a}_{a^*} \tag{4}$$

determines also an element of $\overline{M}$, and also the change due to a displacement from $\overline{M}(a^*)$ to $\overline{M}(b^*)$

$$-\Gamma(\bar{a}_{a^*b^*})=\bar{a}_{b^*}(\bar{a}_{a^*}, D_{a^*b^*})-\bar{a}_{a^*}. \tag{5}$$

The difference

$$\nabla\bar{a}_{a^*b^*}=\Delta\bar{a}_{a^*b^*}+\Gamma(\bar{a}_{a^*b^*}) \tag{6}$$

for any set $b^*$ contained in a neighbourhood of the set $a^*$ is called the *covariant change* of $\bar{a}_{a^*}(a^*)$ at the set $a^*$. The covariant change $\nabla\bar{a}_{a^*b^*}$ determines an element in the manifold $\overline{M}_{b^*}$,

7. Let us consider a one-to-one and continuous transformation in any manifold $\overline{M}_{a^*}$

$$'\bar{a}_{a^*}=f(\bar{a}_{a^*}, a^*), \tag{7}$$

where $f(\bar{a}_{a^*}, a^*)$ is a continuous set-function with respect to $\bar{a}_{a^*}$ and $a^*$. By this transformation, both the changes $\Delta\bar{a}_{a^*b^*}$ and $\Gamma(\bar{a}_{a^*b^*})$ depend upon the variation of $f(\bar{a}_{a^*}, a^*)$ due to the change from $a^*$ to $b^*$, but that of the covariant change does not and

$$'\nabla\bar{a}_{a^*b^*}=\varphi(\nabla\bar{a}_{a^*b^*}, \bar{a}_{b^*}(\bar{a}_{a^*}, D_{a^*b^*}), b^*), \tag{8}$$

where

$$(9)\qquad \varphi(\nabla\bar{a}_{a^*b^*}, \bar{a}_{b^*}(\bar{a}_{a^*}, D_{a^*b^*}), b^*)=f(\bar{a}_{b^*}, b^*)-f(\bar{a}_{b^*}(\bar{a}_{a^*}, D_{a^*b^*}), b^*)\,,$$

so that

$$\begin{aligned} f(\bar{a}_{b^*}, b^*)&=f(\bar{a}_{b^*}(\bar{a}_{a^*}, D_{a^*b^*})+\nabla\bar{a}_{a^*b^*}, b^*)\\ &=f(\bar{a}_{b^*}(\bar{a}_{a^*}, D_{a^*b^*}), b^*)+\varphi(\nabla\bar{a}_{a^*b^*},\ \bar{a}_{b^*}(\bar{a}_{a^*}D_{a^*b^*}), b^*)\,. \end{aligned}$$

When $f(\bar{a}_{a^*}, a^*)$ is linear with respect to $\bar{a}_{a^*}$, namely

$$f(\bar{a}_{a^*}+\bar{b}_{a^*}, a^*)=f(\bar{a}_{a^*}, a^*)+f(\bar{b}_{a^*}, a^*)\,,$$

then the covariant change $\nabla\bar{a}_{a^*b^*}$ is transformed by this transformation in such a manner as $\bar{a}_{a^*}$. This result corresponds to the well-known fact in the ordinary tensor calculus, that the covariant differential of a vector is also a vector.

8. For a convergent sequence $b^{*(1)}, b^{*(2)}, \ldots\ldots$ to the set $a^*$, let a sequence of displacement $D_{a^*b^{*(1)}}, D_{a^*b^{*(2)}}, \ldots\ldots$ tend to the identical displacement $D_{a^*a^*}=1$[1], then

$$\lim_{r\to\infty}\{\bar{a}_{b^{*(r)}}(\bar{a}_{a^*}, D_{a^*b^{*(r)}})-\bar{a}_{a^*}(a^*)\}=0\,.$$

In this case we can write (6) in a form of differentials

$$(10)\qquad \delta\bar{a}_{a^*}=d\bar{a}_{a^*}+\Gamma(\bar{a}_{a^*})\,,$$

which is called the *covariant differential* of $\bar{a}_{a^*}(a^*)$ with respect to the above sequence of displacements.

9. We assume that the displacement $D_{a^*b^*}$ is determined uniquely by a curve $C_{a^*b^*}$ joining $a^*$ and $b^*$[2] and that the manifold $M^*$ is also a space $(K)$. Then

$$\begin{aligned} (11)\qquad -\Gamma(\bar{a}_{a^*b^*})&=\bar{a}_{b^*}(\bar{a}_{a^*}, C_{a^*b^*})-\bar{a}_{a^*}\\ &=-\Gamma(\bar{a}_{a^*},a^*, da^*, d^2a^*, \ldots\ldots) \end{aligned}$$

for $b^*$ approaching to $a^*$[3] on the curve $C_{a^*b^*}$.

When every set on the curve $C_{a^*b^*}$ is represented by a numerical value of a parameter $t$, then we have in general

$$(12)\qquad \delta\bar{a}_{a^*}=d\bar{a}_{a^*}+\Gamma\left(\bar{a}_{a^*}, a^*, \frac{da^*}{dt}, \frac{d^2a^*}{dt^2}, \ldots\ldots\right)dt\,.$$

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1) This means that any neighbourhood of every element $\bar{a}_{a^*}$ in the manifold $\bar{M}_{a^*}$ contains the equal element of $\bar{a}_{b^{*(r)}}(\bar{a}_{a^*}, D_{a^*b^{*(r)}})$ for all $r$ greater than a fixed integer $r_0$ which depends only upon the neighbourhood.

2) A curve joining two sets $a^*$ and $b^*$ in the manifold $M^*$ is a connected continuum containing these sets and having no connected subcontinuum which contains $a^*$ as well as $b^*$.

3) When $b^*-a^*$ for a fixed $a^*$ tends to zero-element, we say $b^*$ approaches to $a^*$.