136. *Remarks on the Cesàro Summability of Divergent Series.*

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The object of this paper is to prove a converse of Cauchy's theorem concerning limit and give alternate proofs of Doetsch's theorem[1] and the well-known Cesàro-Tauberian theorem due to Hardy and Landau.

1. *Theorem* I. *If*

$$na_n \geqq (n-1)a_{n-1}, \qquad n>1, \tag{1}$$

then

$$\lim_{n\to\infty} \frac{a_1+a_2+\cdots\cdots+a_n}{n} = L \tag{2}$$

implies $\lim_{n\to\infty} a_n = L$.

Proof. Since the sequence (na_n) is monotone increasing, its limit exists. If the limit of (na_n) is finite, then $a_n \to 0$, consequently L must be 0. In this case the theorem is evident. If $L \neq 0$, the limit of (na_n) can not be finite. Thus we have to discuss the case, where na_n tends to infinity.

Plainly we can suppose that a_n is positive for all n. For any positive number ε, there is an integer n_0 such that

$$\left| \frac{a_1+a_2+\cdots\cdots+a_n}{n} - L \right| < \varepsilon, \tag{3}$$

for $n \geqq n_0$. Let p be a fixed positive integer, then

$$\frac{a_1+a_2+\cdots\cdots+a_n+a_{n+1}+\cdots\cdots+a_{n+[\frac{n}{p}]}}{n+\left[\frac{n}{p}\right]} < L+\varepsilon,$$

for $n \geqq n_0$, where $[x]$ denotes the integral part of x.

From (3), we have

1) Doetsch: Über die Cesàrosche Summabilität bei Reihen und eine Erweiterung des Grenzwertbegriffs bei integrablen Funktionen. Math. Zeit. **11** (1921). See Nikola Obreschkoff: Über einige Sätz für Summierung von divergenten Reihen. Tôhoku Math. Journ. **32** (1930).

$$(4)\qquad \frac{a_{n+1}+\cdots\cdots+a_{n+[\frac{n}{p}]}}{n+\left[\frac{n}{p}\right]}<\frac{n}{n+\left[\frac{n}{p}\right]}\varepsilon+\varepsilon+\left(1-\frac{n}{n+\left[\frac{n}{p}\right]}\right)L$$

$$<2\varepsilon+\left(1-\frac{n}{n+\left[\frac{n}{p}\right]}\right)L,$$

and from (1), it results

$$(5)\qquad a_{n+q}\geqq\frac{n}{n+q}a_n,$$

for any positive integer q. Putting (5) into (4), we have

$$\left(\frac{n}{n+1}+\cdots\cdots+\frac{n}{n+\left[\frac{n}{p}\right]}\right)a_n\Big/\left(n+\frac{n}{p}\right)<2\varepsilon+\left(1-\frac{n}{n+\left[\frac{n}{p}\right]}\right)L,$$

$$\left(\log\frac{n+\left[\frac{n}{p}\right]}{n}+\gamma_{n+[\frac{n}{p}]}-\gamma_n\right)a_n<\frac{n+\left[\frac{n}{p}\right]}{n}\left\{2\varepsilon+\left(1-\frac{n}{n+\left[\frac{n}{p}\right]}\right)L\right\},$$

where $\gamma_n=1+\frac{1}{2}+\cdots\cdots+\frac{1}{n}-\log n$, which tends to the Euler's constant. If η be an arbitrary positive number, then there exists an integer n_1 such that $\qquad |\gamma_{n+[\frac{n}{p}]}-\gamma_n|<\eta, \qquad$ for $n\geqq n_1$.

Thus we have $\quad \left\{\log\left(1+\frac{1}{p}\right)-\eta\right\}a_n<4\varepsilon+\left(\frac{1}{p}+\frac{1}{n}\right)L,$

for $n\geqq \text{Max}(n_0, n_1)$. Letting $n\to\infty$, we have

$$\overline{\lim_{n=\infty}}\, a_n\cdot\left\{\log\left(1+\frac{1}{p}\right)-\eta\right\}\leqq 4\varepsilon+\frac{1}{p}L.$$

Since ε and η are arbitrary, $\quad \overline{\lim_{n=\infty}}\, a_n\leqq\dfrac{L}{p\log\left(1+\frac{1}{p}\right)}.$

Since p is arbitrary, we have by letting $p\to\infty$

$$(6)\qquad \overline{\lim_{n=\infty}}\, a_n\leqq L.$$

Next, for any positive number ε, there is an n_2 such that

$$L-\varepsilon<\frac{a_1+a_2+\cdots\cdots+a_{[\frac{n}{p}]}+a_{[\frac{n}{p}]+1}+\cdots\cdots+a_n}{n}$$

for $n\geqq p(n_2+1)$, where p is a fixed positive number >1. Hence

$$(7)\qquad L-\varepsilon<\frac{a_1+a_2+\cdots\cdots+a_{[\frac{n}{p}]}}{n}+\frac{a_{[\frac{n}{p}]+1}+\cdots\cdots+a_n}{n}.$$

Putting (5) into (7), we have

$$L-\varepsilon<\left(\frac{a_1+a_2+\cdots\cdots+a_{\left[\frac{n}{p}\right]}}{\left[\frac{n}{p}\right]}-L\right)\frac{\left[\frac{n}{p}\right]}{n}+\left(\frac{n}{\left[\frac{n}{p}\right]+1}+\frac{n}{\left[\frac{n}{p}\right]+2}+\cdots\cdots+\frac{n}{n}\right)\frac{a_n}{n}+L\frac{\left[\frac{n}{p}\right]}{n}$$

$$<\varepsilon\left(\frac{1}{p}+\frac{1}{n}\right)+\left\{\log\frac{n}{\left[\frac{n}{p}\right]}+(\gamma_n-\gamma_{\left[\frac{n}{p}\right]})\right\}a_n+L\left(\frac{1}{p}+\frac{1}{n}\right)$$

$$<\varepsilon\left(\frac{1}{p}+\frac{1}{n}\right)+\left(\log\frac{1}{\frac{1}{p}-\frac{1}{n}}+\eta\right)a_n+L\left(\frac{1}{p}+\frac{1}{n}\right),$$

for $n\geqq \mathrm{Max}\,(p(n_2+1), n_1)$. Letting $n\to\infty$, we have

$$L-\varepsilon\leqq\frac{\varepsilon}{p}+(\log p+\eta)\varliminf_{n=\infty} a_n+\frac{L}{p}.$$

Since ε and η are arbitrary, we have

$$\frac{L\left(1-\frac{1}{p}\right)}{\log p}\leqq\varliminf_{n=\infty} a_n.$$

Letting $p\to 1$, we have

(8) $$L\leqq\varliminf_{n=\infty} a_n.$$

From (7) and (8), we have finally

$$\lim_{n=\infty} a_n=L,$$

which is the required result.

2. *Theorem* II. *Put* $\sum_{\nu=1}^{n} a_\nu=S_n$. *If* $S_n>-k$, *k being a constant, then the fact that* $\sum_1^\infty a_n$ *is* (C, r) *summable* $(r>1)$, *implies that* $\sum_1^\infty a_n$ *is* $(C, 1)$ *summable.*

Proof. Without loss of generality, we can suppose that $r=2$. For our purpose, it is sufficient to prove that if

$$\left\{S_1+\frac{S_1+S_2}{2}+\cdots\cdots+\frac{S_1+S_2+\cdots\cdots+S_n}{n}\right\}\Big/ n\to L,$$

then $$\frac{S_1+S_2+\cdots\cdots+S_n}{n}\to L.$$

We can suppose that $k=0$. For otherwise we take $S_\nu+k$ for S_ν. Then the theorem is evident from Theorem I.

Theorem III. *If* $\sum_1^\infty (na_n-(n-1)a_{n-1})$ *is* (C, r) *summable to* L, r *being positive and* $na_n > -k$, *then* $\dfrac{a_1+2a_2+\cdots\cdots+na_n}{n}$ *tends to* L.

If we take $na_n-(n-1)a_{n-1}$ for a_n in Theorem II ($a_0=0$), then we have Theorem III.

Theorem IV. *The series, which is one-sidedly bounded* (C, r) $(r>-1)$ *and* (C, r') *summable, is* $(C, r+1)$ *summable.*

This theorem is due to Dr. Doetsch.

Proof. If the series is $(C, r+1)$ summable, then the arithmetic mean of (C, r) partial sum tends to a limit. Therefore the theorem is valid by Theorem II, where r is any number greater than -1.

3. *Theorem* V. *If* $\sum_1^\infty a_n$ *is* (C, r) *summable and* $na_n>-k$, *then* $\sum_1^\infty a_n$ *converges.*

This is the Hardy-Landau's theorem.

Proof. We can suppose that r is an integer. Let

$$T_n^{(0)}=\sum_1^n a_\nu\,,$$

$$T_n^{(1)}=\sum_1^n T_\nu^{(0)}\,,\qquad \tau_n^{(1)}=\frac{T_n^{(1)}}{n}\,,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$T_n^{(r)}=\sum_1^n \tau_\nu^{(r-1)}\,,\qquad \tau_n^{(r)}=\frac{T_n^{(r)}}{n}\,,\qquad (r>1)\,,$$

and

$$U_n^{(0)}=\sum_1^n (\nu a_\nu-(\nu-1)a_{\nu-1})\,,$$

$$U_n^{(1)}=\sum_1^n \nu a_\nu\,,\qquad u_n^{(1)}=\frac{U_n^{(1)}}{n}\,,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$U_n^{(r)}=\sum_1^n u_n^{(r-1)}\,,\qquad u_n^{(r)}=\frac{U_n^{(r)}}{n}\,,\qquad (r>1)\,.$$

Then we have $U_n^{(1)}=nT_n^{(0)}-T_n^{(1)}$, consequently $u_n^{(1)}=T_n^{(0)}-\tau_n^{(1)}$ and in general $u_n^{(r+1)}=\tau_n^{(r)}-\tau_n^{(r+1)}$.

If $\sum_1^\infty a_n$ is (C, r) summable, then $\tau_n^{(r)}$ tends to a limit and hence $u_n^{(r+1)}$ tends to zero. Therefore $\sum_1^\infty (na_n-(n-1)a_{n-1})$ is $(C, r+1)$ summable to zero. Hence by Theorem III, we have

$$\frac{a_1+2a_2+\cdots\cdots+na_n}{n}\to 0\,.$$

Consequently $\sum_1^\infty a_n$ is convergent. Thus the theorem is proved.