163. A New Concept of Integrals.

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The object of this paper is to define the integral, which is more general than the Ridder's integral, and then than those of Danjoy and Burkill.

1. Let $f(x)$ be defined in an interval (a, b). If we can find a set E, such that

1°. x is the point of exterior density of E,

2°. $$\lim_{\xi(\varepsilon E)\to x} \frac{f(\xi)-f(x)}{\xi - x} \tag{1}$$

exists and is finite, where the limit is taken such that ξ tends to x, lying in E, then $f(x)$ is called to be *approximately differentiable* at x, and the value (1) is called the *approximate derivative* at x and is denoted by $ADf(x)$.

Theorem 1. If $f(x)$ is measurable and is approximately differentiable at x, then there is a measurable set E, such that

1°. x is the point of density of E,

2°. $\lim\limits_{\xi(\varepsilon E)\to x} \dfrac{f(\xi)-f(x)}{\xi - x}$ exists and is equal to $ADf(x)$.

2. If we can find a set E, such that

1°. the inferior interior density of E at x is $\geqq \tau (>0)$,

2°. $$\lim_{\xi(\varepsilon E)\to x} \frac{f(\xi)-f(x)}{\xi - x} \tag{2}$$

exists and is finite, then $f(x)$ is called to be (τ)-*approximately differentiable* at x. The value (2) is called the (τ)-*approximate derivative* of $f(x)$ at x, and is denoted by $\underset{\tau}{A}Df(x)$. And put $\underset{1}{A}Df(x) = A^*Df(x)$.

Next, let E be a set such that the inferior interior density on the right hand of E at x is $\geqq \tau (>0)$.

Put $$a_E = \overline{\lim_{\xi(\varepsilon E)\to x}} \frac{f(\xi)-f(x)}{\xi - x}.$$

The lower bound of a_E is defined to be the *upper (τ)-approximate derivative on the right hand* of $f(x)$ at x, and denoted by $\underset{\tau}{A}D^+f(x)$. When we do not concern with the value of τ, we denote it by $AD^{+}f(x)$. When $\tau=1$, we denote it by $A^{*}D^{+}f(x)$.

Similarly, $\underset{\tau}{A}D_{+}f(x)$, $\underset{\tau}{A}D^{-}f(x)$ and $\underset{\tau}{A}D_{-}f(x)$ are defined.

Theorem 2. If $\tau>0$, then $\underset{\tau}{A}\overline{D}f(x)\geqq\underset{\tau}{A}\underline{D}f(x)$.

Theorem 3. If $\tau>0$, then $\underset{\tau}{A}\overline{D}f(x)=-\underset{\tau}{A}\underline{D}\{-f(x)\}$.

Theorem 4. If $\tau>\frac{1}{2}$, then $\underset{2\tau-1}{A}\overline{D}\{f(x)+g(x)\}\leqq\underset{\tau}{A}\overline{D}f(x)+\underset{\tau}{A}\overline{D}g(x)$.

Theorem 5. If $\tau>\frac{1}{2}$, then $\underset{2\tau-1}{A}\underline{D}\{f(x)+g(x)\}\geqq\underset{\tau}{A}\underline{D}f(x)+\underset{\tau}{A}\underline{D}g(x)$.

Theorem 6. The necessary and sufficient condition that $f(x)$ is (τ)-approximately differentiable, is that $\underset{\tau}{A}\overline{D}f(x)=\underset{\tau}{A}\underline{D}f(x)$.

3. Suppose that $f(x)$ is defined in (a, b). If, for any ε (>0) and any x ($a<x<b$), there is a set E, such that

1°. x is the point of exterior density of E,

2°. $|f(\xi)-f(x)|<\varepsilon$ for all ξ in E,

then we say that $f(x)$ is *approximately continuous* at x.

If $x=a$ or $x=b$, then $f(x)$ is said to be approximately continuous, when x is the point of exterior density on the right (or left) hand of E and 2° is satisfied.

Theorem 7. If a measurable function $f(x)$ is approximately continuous at x, then there is a measurable set E, such that

1°. x is the point of density of E,

2°. $\lim\limits_{\xi(\varepsilon E)\to x} f(\xi)=f(x)$.

If, for any ξ (>0) and any x ($a<x<b$), we can find a set E, such that

1°. the inferior interior density of E at x is $\geqq\tau(>0)$.

2°. $|f(\xi)-f(x)|<\varepsilon$ for all ξ in E,

then we say that $f(x)$ is (τ)-*approximately continuous*[1] at x.

Theorem 8. If $f(x)$ is defined and is finite almost everywhere in (a, b), then $f(x)$ is almost everywhere approximately differentiable in the set $E(\underset{\tau}{A}D^{+}f(x)<+\infty)$.

4. *Theorem 9.* Suppose that $\tau>\frac{1}{2}$ and $f(x)$ is defined and is everywhere (τ)-approximately continuous in the closed interval $[a, b]$. If $\underset{\tau}{A}D_{+}f(x)\geqq 0$, with the possible exception of enumerable set in (a, b), then $f(x)$ is non-decreasing in (a, b). Specially, $f(b)\geqq f(a)$.

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1) (τ)-approximately continuous function is measurable. I owe this remark to Prof. J. Ridder.

5. Let $f(x)$ be almost everywhere finite in (a, b). $M(x)$ is called a *major function* of $f(x)$ in (a, b), if it satisfies the following conditions:

1°. $M(x)$ is (τ)-approximately continuous in the closed interval $[a, b]$, $(\tau > \frac{1}{2})$,

2°. $M(a) = 0$,

3°. $\underset{\tau}{AD}M(x) > -\infty$ with the exception of enumerable set in (a, b),

4°. $\underset{\tau}{AD}M(x) \geqq f(x)$ for all x in (a, b).

Similarly, a *minor function* $m(x)$ is defined. $M(x)$ and $m(x)$ are called *associated functions* of $f(x)$ in (a, b).

Theorem 10. If $f(x)$ is defined in (a, b), and $M(x)$ and $m(x)$ are the associated functions of $f(x)$, then $M(x)-m(x)$ is a positive non-decreasing function. In particular, $M(b) \geqq m(b)$.

Suppose that $f(x)$ is defined and is almost everywhere finite in (a, b), and the associated functions $M(x)$ and $m(x)$ of $f(x)$ exist.

If we put $I_1(b) =$ lower bound of all $M(b)$
and $I_2(b) =$ upper bound of all $m(b)$,
then they are finite and $I_1(b) \geqq I_2(b)$.

If $I_1(b) = I_2(b)$, then $f(x)$ is said to be (τ)-*integrable* in (a, b), and the value $I_1(b)$ is called the (τ)-*integral*, and is denoted by

$$(\tau)\int_a^b f(x)dx .$$

6. *Theorem 11.* If $f(x)$ is (τ)-integrable in (a, b), then $f(x)$ is also (τ)-integrable in any subinterval of (a, b).

Theorem 12. If $a<b<c$ and $f(x)$ is (τ)-integrable in (a, b) and (b, c), then $f(x)$ is (τ)-integrable in (a, c) and

$$(\tau)\int_a^c f(x)dx = (\tau)\int_a^b f(x)dx + (\tau)\int_b^c f(x)dx .$$

Theorem 13. If $f(x)$ is (τ)-integrable in (a, b) and c is a constant, then

$$(\tau)\int_a^b \{c \cdot f(x)\}dx = c \cdot (\tau)\int_a^b f(x)dx .$$

Theorem 14. If $f_1(x)$ and $f_2(x)$ are (τ)-integrable, then $f_1(x)+f_2(x)$ is (σ) integrable, and

$$(\tau)\int_a^b f_1(x)dx + (\tau)\int_a^b f_2(x)dx = (\sigma)\int_a^b \{f_1(x)+f_2(x)\}dx ,$$

where $\sigma = 2\tau - 1$.

Theorem 15. If $f_1(x)$ and $f_2(x)$ are (τ)-integrable, and $f_1(x) \geqq f_2(x)$, then

$$(\tau)\int_a^b f_1(x)dx \geqq (\tau)\int_a^b f_2(x)dx , \qquad \text{where } \tau > \tfrac{2}{3}.$$

Theorem 16. The indefinite integral $F(x) = (\tau)\int_a^x f(t)dt$ $(a \leqq x \leqq b)$ is a (τ)-approximately continuous function of x.

Theorem 17. If $F(x)=(\tau)\int_a^x f(t)dt$, then $\underset{\tau}{A}DF(x)=f(x)$ for almost all x in (a, b).

Theorem 18. If $f(x)$ is non-negative in (a, b), then $f(x)$ is (τ)-integrable and integrable in Lebesgue's sense at the same time, having the same value.

Theorem 19. If $\{f_n(x)\}$ is a sequence of (τ)-integrable functions, such that

1°. $\lim_{n\to\infty} f_n(x)=f(x)$,

2°. there is a (τ)-integrable function $g(x)$, such that

$$|f_n(x)|\leqq g(x) \qquad (n=1, 2, 3, \ldots\ldots),$$

then $f(x)$ is (τ)-integrable, and $\lim_{n\to\infty}(\tau)\int_a^b f_n(x)dx=(\tau)\int_a^b f(x)dx$.

Theorem 20. If $f(x)$ is (τ)-integrable, then $f(x)$ is measurable.