# 16. A New Concept of Integrals, II.[1]

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7. Let $f(x)$ be almost everywhere finite in $(a, b)$. $M^*(x)$ is called a *major* function* of $f(x)$ in $(a, b)$, if it satisfies the following conditions:

1°. $M^*(x)$ is $(\tau)$-approximately continuous in the closed interval $[a, b]$, $(\tau > \frac{1}{2})$.

2°. $M^*(a) = 0$.

3°. $(a, b)$ is covered by a system of enumerable perfect sets $\{P_i\}$, except an enumerable set at most, such that

3°. 1. $\underset{\tau}{A}\underline{D}M_i(x) > -\infty$

with the possible exception of an enumerable set in $P_i$,

3°. 2. $\underset{\tau}{A}\underline{D}M_i(x) \geqq f(x)$

with the possible exception of an enumerable set in $P_i$, where $M_i(x)$ is defined such that

$$M_i(x) = M^*(x), \text{ for } x \text{ in } P_i \text{ and for } x=a,\ x=b,$$

and $M_i(x)$ is linear in the contiguous intervals of $P_i$.

4°. For any perfect subset $Q_i$ of $P_i$, $N_i(x)$, defined as $M_i(x)$, taking $Q_i$ instead of $P_i$, has the corresponding properties of $M_i(x)$.

Similarly, a *minor* function* $m^*(x)$ is defined. $M^*(x)$ and $m^*(x)$ are called the associated* functions of $f(x)$ in $(a, b)$.

*Theorem 21.* If $f(x)$ is defined in $(a, b)$, and $M^*(x)$ and $m^*(x)$ are the associated* functions of $f(x)$, then $M^*(x) - m^*(x)$ is a positive non-decreasing function. In particular,

$$M^*(b) \geqq m^*(b).$$

Suppose that $f(x)$ is defined and is almost everywhere finite in $(a, b)$, and the associated* functions $M^*(x)$ and $m^*(x)$ of $f(x)$ exist.

If we put

$$I_1^*(b) = \text{lower bound of all } M^*(b),$$

and

$$I_2^*(b) = \text{upper bound of all } m^*(b),$$

then they are finite and

1) In the first paper (this volume, No. 10, pp. 570–574), I have to correct the following points: 1°. In Theorem 2 and 3, read $\tau > \frac{1}{2}$ for $\tau > 0$; 2°. To the last of Theorem 6, add $(\tau > \frac{1}{2})$.

$$I_1^*(b) \geqq I_2^*(b) .$$

If $I_1^*(b)=I_2^*(b)$, then $f(x)$ is said to be $(\tau^*)$-*integrable* in $(a, b)$, and the common value $I_1^*(b)$ is called the $(\tau^*)$-*integral*, and is denoted by

$$(\tau^*)\int_a^b f(x)dx .$$

8. Omitting the obvious properties of $(\tau^*)$-integrals, we get the following theorems.

*Theorem 22.* If $f(x)$ is $(\tau)$-integrable in $(a, b)$, then $f(x)$ is $(\tau^*)$-integrable.

*Theorem 23.* If $\frac{1}{2}<\tau_1<\tau_2\leqq 1$ and $f(x)$ is $(\tau_2^*)$-integrable, then $f(x)$ is $(\tau_1^*)$-integrable. Particularly, if $f(x)$ is $(S)$-integrable (in the Ridder's sense), then $f(x)$ is $(\tau_1^*)$-integrable.

*Theorem 24.* If $f_1(x)$ and $f_2(x)$ are $(\tau^*)$-integrable, then $f_1(x)+f_2(x)$ is $(\sigma^*)$-integrable, and

$$(\sigma^*)\int_a^b \{f_1(x)+f_2(x)\}dx=(\tau^*)\int_a^b f_1(x)dx+(\tau^*)\int_a^b f_2(x)dx ,$$

where $\tau>\frac{3}{4}$ and $\sigma=2\tau-1$.

*Theorem 25.* If $f_1(x)$ and $f_2(x)$ are $(\tau^*)$-integrable, and $f_1(x)\geqq f_2(x)$, then

$$(\tau^*)\int_a^b f_1(x)dx\geqq(\tau^*)\int_a^b f_2(x)dx ,$$

where $\tau>\frac{2}{3}$.

*Theorem 26.* The indefinite integral $F(x)=(\tau^*)\int_a^x f(t)dt$ $(a\leqq x\leqq b)$ is a $(\tau)$-approximately continuous function of $x$.

*Theorem 27.* If $F(x)=(\tau^*)\int_a^x f(t)dt$, then

$$\underset{\tau}{AD}F(x)=f(x)$$

for almost all $x$ in $(a, b)$.

*Theorem 28.* If $f(x)$ is non-negative in $(a, b)$, then $f(x)$ is $(\tau^*)$-integrable and integrable in Lebesgue's sense at the same time, having the same value.

*Theorem 29.* If $\{f_n(x)\}$ is a sequence of $(\tau^*)$-integrable functions, such that

1°. $\lim_{n\to\infty} f_n(x)$ exists and $=f(x)$,

2°. there is a $(\tau^*)$-integrable function $g(x)$

such that $\quad |f_n(x)|\leqq g(x) \qquad (n=1, 2, 3, \ldots\ldots)$,

then $f(x)$ is $(\tau^*)$-integrable, and

$$\lim_{n\to\infty}(\tau^*)\int_a^b f_n(x)dx=(\tau^*)\int_a^b f(x)dx .$$