14. *Kinematic Connections and Their Application to Physics.*

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Recently a new physical theory has been developed by O. Veblen,[1] J. A. Schouten[2] and others in which the principal point is founded on a projective connection. In the present paper we shall develop some connections in the manifold admitting the kinematic transformations, and shall give a unification of the gravitational field not only with the electromagnetic, but also with Dirac's theory of material waves.

Let the equations

$$\text{(1. a)} \qquad \bar{x}^i=\bar{x}^i(x^0, x^1, x^2, x^3, x^4)\,, \qquad i=1, 2, 3, 4,$$

be the transformations of the coördinates in X_4, where x^0 is a parameter, and we shall define the transformation of the parameter by

$$\text{(1. b)} \qquad \bar{x}^0=x^0\,.$$

These transformations (1. a) and (1. b) are collectively called a *kinematic transformation* in the manifold X_4.

The kinematic transformation (1. a), (1. b) can be regarded as follows. An ordered set of the five independent real variables x^ν $(\nu=0, 1, 2, 3, 4)$,[3] of which at least one is not zero may be considered as a coördinate system of a 5-dimensional manifold X_5 except the original point. Two points x^ν and y^ν are called coincident if a factor exists, so that $y^\nu=\sigma x^\nu$. Each totality of all points coincident with any point is called a spot. The totality of all ∞^4 spots is called the 4-dimensional projective manifold P_4. The set of all points of the P_4, with the exception of those on a single 3-dimensional projective manifold P_3 contained in the P_4, is called the affine manifold A_4. By choosing the P_3 as the hyperplane at infinity, the equation of the P_3 may be written in the form $x^0=0$. Thus (1. a) and (1. b) are transformations of coördinates in A_4, and by them P_3 is transformed into itself.

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1) O. Veblen: Projektive Relativitätstheorie. Julius Springer, 1933.

2) J. A. Schouten und D. van Dantzig: Generelle Feldtheorie, Zeit. für Physik, **78** (1932), 639–667.

3) Let us make the convention that Greek indices run over the range 0, 1, 2, 3, 4, whereas the Latin indices take on the values 1, 2, 3, 4 only.

If V^α and $\bar{V}^\alpha$ are functions of the x's and $\bar{x}$'s respectively such that

$$(2) \qquad \bar{V}^0 = V^0, \qquad \bar{V}^i = \frac{\partial \bar{x}^i}{\partial x^j} V^j + \frac{\partial \bar{x}^i}{\partial x^0} V^0$$

in consequence of (1), V^α and $\bar{V}^\alpha$ are the components of a *kinematic contravariant vector* in the coördinate systems (x) and $(\bar{x})$ respectively. A *kinematic covariant vector* is a set of the quantities W_α which is transformed by (1) into

$$(3) \qquad \bar{W}_0 = W_0 + \frac{\partial x^i}{\partial \bar{x}^0} W_i, \qquad \bar{W}_i = \frac{\partial x^j}{\partial \bar{x}^i} W_j.$$

A similar observation is applied to the *kinematic tensors* of the higher order.[1]

With any point (x^1, x^2, x^3, x^4) of X_4 there is associated a tangential space $E_4(dx^1, dx^2, dx^3, dx^4)$. The point $dx^i=0$ is identified with the point x^i and will be called the point of contact. These tangential spaces can be improved into ordinary projective spaces $\bar{E}_4$ by introducing in each of them a hyperplane $\bar{E}_3$ at infinity in the usual manner.

Let a fixed value ξ of the parameter x^0 correspond to a point P (x^1, x^2, x^3, x^4) of the X_4. Then in a neighbourhood of the point $(\xi, x^1, x^2, x^3, x^4)$ we shall introduce a 5-dimensional euclidean space E_5, having $(\xi, x^1, x^2, x^3, x^4)$ as origin. In particular we assume that the coördinates in E_5 are connected by the formulas $X^0=dx^0$, $X^i=dx^i$. Then the point $dx^0=0$ and $dx^i=0$ is the original point in the E_5.

Let us choose a tangential projective space $\bar{E}_4$ at the point, whose coördinates are $X^i=0$, $X^0=dx^0$ in E_5. Then each of the straight lines through the origin of E_5 cuts $\bar{E}_4$ in one and only one point. The coördinates of the point (X^0, X^i) can be regarded as the homogeneous coördinates for the points of $\bar{E}_4$.

In every local tangential projective space $\bar{E}_4$ we introduce a non-degenerate quadric $G^{\alpha\beta}U_\alpha U_\beta=0$, which does not pass through the contact point (1, 0, 0, 0, 0), where U's are the hyperplane coördinates in $\bar{E}_4$. The quadric is determined uniquely by a symmetric kinematic tensor $G^{\alpha\beta}$. Hence in each local $\bar{E}_4$ we can consider a non-euclidean geometry, by introducing the quadric as the absolute. The envelope of all hyperplanes meeting a hyperplane $[U_0=1, U_i=0]$ at a constant angle ω is a hypersphere, specially the equation of the hypersphere having the angle $\omega=0$ is given by the equation

1) T. Hosokawa: Tôkyo Butsuri-gakko Zasshi, 42, No. 500 (July, 1933), p. 376–382. Since this paper was completed, the author has seen the same definition used by V. Hlavatý: Über eine Art der Punktkonnexion, Math. Zeit. **38** (1933), 135–145.

(4) $$\{G^{\alpha\beta}-(G^{0\alpha}G^{0\beta})/G^{00}\}U_\alpha U_\beta=0\,.$$

This hypersphere touches the absolute at the curve of intersection of the absolute with a *definite hyperplane*

(5) $$G^{0\alpha}U_\alpha=0\,.$$

Putting

$$\frac{G^{\alpha\beta}}{G^{00}}-\frac{G^{0\alpha}G^{0\beta}}{G^{00}G^{00}}=g^{\alpha\beta}\,,$$

we see that $g^{0\alpha}=0$, and that this quadric (4) may be written $g^{ij}U_iU_j=0$.

Let us denote by $|g|$ the determinant of the g^{ij}'s, by g_{jk} the cofactors of g^{jk} divided by $|g|$, then we have $g^{ij}g_{jk}=\delta^i_k$. So that under a pure transformation of coördinates

(6) $$\bar{x}^0=x^0=\text{const.},\qquad \bar{x}^i=\bar{x}^i(x^1,x^2,x^3,x^4)\,,$$

the components g_{ij} are transformed like components of an arbitrary tensor. Then g_{ij} may be regarded as the fundamental tensor of a Riemannian space.

Putting also $G^{0\alpha}/G^{00}=\varphi^\alpha$, we get $\varphi^\alpha U_\alpha=0$ from (5), as the equation of a definite hyperplane. Then φ^α is a contravariant vector and $\varphi^0=1$, and under a transformation (6) the components φ^i are transformed in the form

$$\bar{\varphi}^i=\frac{\partial\bar{x}^i}{\partial x^j}\varphi^j\,.$$

We shall interpret the coefficients g_{ij} and vectors φ_i as the gravitational and electromagnetic potentials respectively, where $\varphi_i=g_{ij}\varphi^j$.

Let us now put $(G^{00})^{\frac{1}{2}}=\phi$, then we obtain $G^{\alpha\beta}=\phi^2(g^{\alpha\beta}+\varphi^\alpha\varphi^\beta)=\phi^2\gamma^\alpha$, where $\gamma^{\alpha\beta}=g^{\alpha\beta}+\varphi^\alpha\varphi^\beta$. Let $\gamma_{\alpha\beta}$ be defined by the equation $\gamma^{\alpha\beta}\gamma_{\beta\delta}=\delta^\alpha_\delta$, then we get

$$\gamma_{ij}=g_{ij}\,,\qquad \gamma_{00}=1+g_{ij}\varphi^i\varphi^j\,,\qquad \gamma_{0i}=-g_{ij}\varphi^j\,.$$

We will define the connections of the contravariant and covariant vector by the following equations:

$$\nabla_\mu V^\nu=\partial_\mu V^\nu+\Gamma^\nu_{\lambda\mu}V^\lambda \quad\text{and}\quad \nabla_\mu W_\lambda=\partial_\mu W_\lambda-\Gamma^\nu_{\lambda\mu}W_\nu\,.$$

The covariant derivatives $\nabla_\mu V^\nu$ are the components of a mixed tensor of the second order. Hence for the transformation (1), $\bar{\Gamma}^\nu_{\lambda\mu}$ and $\Gamma^\nu_{\lambda\mu}$ must satisfy the equations

$$\bar{\Gamma}^\gamma_{\alpha\beta}\frac{\partial x^\lambda}{\partial\bar{x}^\gamma}=\frac{\partial^2x^\lambda}{\partial\bar{x}^\alpha\partial\bar{x}^\beta}+\Gamma^\lambda_{\mu\nu}\frac{\partial x^\mu}{\partial\bar{x}^\alpha}\frac{\partial x^\nu}{\partial\bar{x}^\beta}\,.$$

We will now define the parameters $\Gamma^\lambda_{\mu\nu}$ by the following expressions:

$$\Gamma^{\lambda}_{\mu\nu}=\frac{1}{2}\gamma^{\lambda\sigma}\left(\frac{\partial\gamma_{\nu\sigma}}{\partial x^{\mu}}+\frac{\partial\gamma_{\sigma\mu}}{\partial x^{\nu}}-\frac{\partial\gamma_{\mu\nu}}{\partial x^{\sigma}}\right),$$

then the equations $\nabla_{\mu}\gamma^{\lambda\nu}=0$ are satisfied identically.

We introduce the hypercomplex numbers of Dirac a^{λ} defined by the equations $a^{(\lambda}a^{\mu)}=G^{\lambda\mu}$, $(a^{\lambda}a^{\mu})a^{\nu}=a^{\lambda}(a^{\mu}a^{\nu})$, $a^{0}=a^{1}a^{2}a^{3}a^{4}$, and consider a local spin-space in each local $\overline{E}_{4}$. Then each a^{λ} may be regarded as a contra- or covariant spinor with valence 2 and may now be written $a^{\lambda A}{}_{B}$ $(A, B, C, D=5, 6, 7, 8)$. If $\Lambda^{A}_{B\mu}$ are the parameters of the covariant differentiation of the contravariant spin-vectors in space-time, then we obtain the Dirac-equation

$$\frac{h}{i}a^{\lambda}\nabla_{\lambda}\psi^{A}=0\,.$$