PAPERS COMMUNICATED

13. *Displacements in a Manifold of Matrices, I.*

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In a previous paper I mentioned some linear displacements in a manifold of matrices.[1] The present paper is devoted to a study of the most general displacements in a manifold of a matrices and of their various special cases. Applications of this theory to geometry and to the theory of spinors in quantum mechanics will be considered in succeeding papers.

1. Let us define the covariant differential of a matrix $A=((a_\mu^\lambda))$ along some direction in the manifold of matrices by

$$\nabla A = dA + \Gamma(A)\,, \tag{1}$$

where $\Gamma(A)$ is a matrix whose elements are homogeneous functions of the first dimension with respect to the matrix A, i.e.

$$\Gamma(A) = \frac{1}{\rho}\Gamma(\rho A) \tag{2}$$

for any quantity ρ. By the transformation of matrix $\bar{A}=\rho A$, i.e. by the variation of the weight of the matrix A, $\Gamma(A)$ let be transformed into

$$\bar{\Gamma}(A) = A d\log\rho + \Gamma(A)\,, \tag{3}$$

then it follows immediately

$$\bar{\nabla}\bar{A} = \frac{1}{\rho}\nabla\rho A\,. \tag{4}$$

2. We consider

$$\Gamma_A \equiv \frac{\partial\Gamma(A)}{\partial A} = \left(\left(\frac{\partial\Gamma(A)}{\partial A_N^M}\right)\right), \tag{5}$$

which is a matrix whose elements $\dfrac{\partial\Gamma(A)}{\partial A_N^M}$ are also matrices, where the row and the column are interchanged: $(\Gamma_A)_M^N = \dfrac{\partial\Gamma(A)}{\partial A_N^M}$. Making use of the notation

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1) A. Kawaguchi: The foundation of the theory of displacements, III, Proc. **10** (1934), 133–136.

$$\Gamma_A \cdot A = \frac{\partial \Gamma(A)}{\partial A^M_N} A^M_N, \tag{6}$$

we have

$$\Gamma_A \cdot A = \Gamma(A), \qquad \Gamma_{AA} \cdot A = 0, \tag{7}$$

by which (1) can be written in the form

$$\nabla A = dA + \Gamma_A \cdot A. \tag{8}$$

Especially, when the matrix is reduced to a unit matrix E, then (8) becomes

$$\nabla E = \Gamma_{\dot{E}}, \tag{9}$$

where $\Gamma_{\dot{E}}$ denotes the norm of the matrix $\left(\left(\frac{\partial \Gamma}{\partial E^M_N}\right)\right)$.

3. We draw into consideration a group of transformations of matrices $\mathfrak{G}$, whose one element is

$$\bar{A} = VAW, \tag{10}$$

V and W being both square matrices. We call such a matrix a contravariant *matrixor* of order one, which depends on the matrix A, and undergoes exactly the same transformation as matrix A:

$$\bar{M} = VM(A)W. \tag{11}$$

By the assumption that the parameter matrix of the displacement Γ is transformed by this transformation (10) as follows:

$$\begin{aligned} \bar{\Gamma}(\bar{A}) &= V\Gamma(A)W - dVAW - VAdW \\ \Gamma(A) &= V^{-1}\bar{\Gamma}(\bar{A})W^{-1} + V^{-1}dVA + AdWW^{-1}, \end{aligned} \tag{12}$$

it follows

$$\bar{\nabla}\bar{A} = V\nabla AW, \tag{13}$$

that is, the matrix $\bar{\nabla}\bar{A}$ is also a matrixor of order one.

4. Let $A \circ B$ denote the matrix whose elements are also matrices $((A^\lambda_\mu B^N_M))_{(\lambda M)(N\mu)}$ for which λ and M are fixed. Then we have the rule of change of the parameter matrices differentiated by A Γ_A:

$$\begin{aligned} \bar{\Gamma}_{\bar{A}} &= (V \times W^{-1})\Gamma_A(W \times V^{-1}) - dVV^{-1} \circ E - E \circ W^{-1}dW, \\ \Gamma_A &= (V^{-1} \times W)\bar{\Gamma}_{\bar{A}}(W^{-1} \times V) + V^{-1}dV \circ E + E \circ dWW^{-1}, \end{aligned} \tag{14}$$

where $V \times W$ denotes the general product of the matrices $((V^\lambda_\mu W^N_M))$ and not $((V^\lambda_\mu W^\mu_M))$. Making use of the elements of the matrices, (14) will be written in the form

$$\begin{aligned} \bar{\Gamma}^{\lambda M}_{\mu N} &= V^\lambda_\alpha W^{-1 M}_{\ \ L} \Gamma^{\alpha L}_{\beta K} W^\beta_\mu V^{-1 K}_{\ \ N} - dV^\lambda_L V^{-1 K}_{\ \ N} E^M_\mu - E^\lambda_N W^{-1 M}_{\ \ K} dW^K_\mu, \\ \Gamma^{\lambda M}_{\mu N} &= V^{-1 \lambda}_{\ \ \alpha} W^M_L \bar{\Gamma}^{\alpha L}_{\beta K} W^{-1 \beta}_{\ \ \mu} V^K_N + E^\lambda_N dW^M_\nu W^{-1 \nu}_{\ \ \mu} + V^{-1 \lambda}_{\ \ \nu} dV^\nu_N E^M_\mu, \end{aligned} \tag{15}$$

being

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(16) $$\Gamma_A = ((\Gamma^{\lambda M}_{\mu N})).$$

5. From (15) we see easily

(17) $$\bar{\Gamma}_{\bar{A}\bar{A}} = (V \times W^{-1} \times W^{-1})\Gamma_{AA}(W \times V^{-1} \times V^{-1}).$$

We call r-ple matrix $M^{(r)} = ((M^{\lambda_1\lambda_2\cdots\cdots\lambda_r}_{\mu_1\mu_2\cdots\cdots\mu_r}))$ a *contravariant* resp. *covariant matrixor of order r*, when it is transformed by the transformation (10) according to the equation

(18) $$\bar{M}^{(r)} = (\underbrace{V \times V \times \cdots\cdots \times V}_{r})M^{(r)}(\underbrace{W \times W \times \cdots\cdots \times W}_{r}) \quad \text{resp.}$$
$$\bar{M}^{(r)} = (\underbrace{W^{-1} \times W^{-1} \times \cdots\cdots \times W^{-1}}_{r})M^{(r)}(\underbrace{V^{-1} \times V^{-1} \times \cdots\cdots \times V^{-1}}_{r})$$

and the *mixed matrixor of order r, of covariant order r−s and of contravariant order s,* if

(19) $$\bar{M}^{(r)} = (V \times V \times \cdots\cdots \times V \times W^{-1} \times W^{-1} \times \cdots\cdots \times W^{-1})$$
$$M^{(r)}(W \times W \times \cdots\cdots \times W \times V^{-1} \times V^{-1} \times \cdots\cdots \times V^{-1}).$$

Then we see that Γ_{AA} is a mixed matrixor of order 3, which is of covariant order two and of contravariant order one.

Put

(20) $$\Gamma_{\cdot AA} = ((\Gamma^{\lambda\mu L}_{\mu\lambda M})),$$

then we have

(21) $$\bar{\Gamma}_{\cdot \bar{A}\bar{A}} = W^{-1}\Gamma_{\cdot AA}V^{-1},$$

which shows that $\Gamma_{\cdot AA}$ is a contravariant matrixor of order one.

6. The quantity $\Gamma_{\cdot A} = \Gamma^{\lambda\mu}_{\mu\lambda}$ changes by the transformation (10) into

(22) $$\bar{\Gamma}_{\cdot \bar{A}} = \Gamma_{\cdot A} - dV \cdot V^{-1} - W^{-1} \cdot dW$$
$$= \Gamma_{\cdot A} - d\log|V| - d\log|W|,$$

where $|V|$ denotes the determinant of the square matrix V.

7. We shall proceed to discuss a special case $VW=E$. In this case the difference between contravariant and covariant vanishes. In fact, the inverse transformation of (10) is

$$\bar{\bar{M}} = WMV,$$

which we can write

$$\overset{*}{\bar{M}} = V\overset{*}{M}W,$$

$\overset{*}{M}$ being the matrix, which we obtain from M by the interchanging rows and columns. For this reason we cannot recognize the difference between contravariant and covariant, for example, (17) becomes in this case

$$\bar{\Gamma}_{\bar{A}\bar{A}} = (V \times V \times V)\Gamma_{AA}(W \times W \times W).$$

The following results are obtained easily.

(I) The norm of matrixor of order one is invariant under the group. More generally, the norm of a matrixor of order r is a matrixor of order $r-1$.

(II) Let A and B be both matrixors of order one, then $A \cdot B$ is invariant under the group and AB is also a matrixor of order one.

(III) $\Gamma_{\cdot A}$ is an invariant under the group.

(IV) $\Gamma_A^{\cdot} = \Gamma_A \cdot E$ is a matrixor of order one.

For
$$\begin{aligned} \overline{\Gamma}{}^{\lambda N}_{\mu N} &= V^{\lambda}_{\alpha} \Gamma^{\alpha N}_{\beta N} W^{\beta}_{\mu} - dV^{\lambda}_{L} V^{-1L}_{\ \ \mu} - W^{-1\lambda}_{\ \ K} dW^{K}_{\mu} \\ &= V^{\lambda}_{\alpha} \Gamma^{\alpha N}_{\beta N} W^{\beta}_{\mu} - dV^{\lambda}_{L} W^{L}_{\mu} - V^{\lambda}_{L} dW^{L}_{\mu} \\ &= V^{\lambda}_{\alpha} \Gamma^{\alpha N}_{\beta N} W^{\beta}_{\mu} . \end{aligned}$$

(V) $\dot{\Gamma}_A$ is a matrixor of order one.

Put

$$(\Gamma_A) = \frac{1}{2}(\Gamma^{\lambda M}_{\mu N} + \Gamma^{M\lambda}_{N\mu}) , \tag{23}$$

then we have

$$(\overline{\Gamma}_A) = (V \times V)(\Gamma_A)(W \times W) ,$$

from which follows

(VI) (Γ_A) is a matrixor of order two.

The theories of the linear transformation group, of the manifold of mixed tensors of the second order, and of spinors all belong to this case.

8. In the theories of the manifold of contravariant or covariant tensors and of quadratic forms, it is $V = W$. In this case the contravariant resp. covariant matrixor of order one M transforms according to the equation

$$\overline{M} = VMV \qquad \text{resp.} \qquad \overline{M} = V^{-1}MV .$$