52. Displacements in a Manifold of Matrices, II.

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As a continuation of the previous paper[1] we introduce in the present paper some parameter matrices of displacement, which are invariant under the weight variations of a matrix, to which the displacement is to be applied.

1. The weight variation of a matrix $\bar{A}=\rho A$ transforms the parameter matrix $\Gamma(A)$ into

$$\bar{\Gamma}(A)=A\varphi+\Gamma(A)\,, \tag{1}$$

taking $\varphi=d\log\rho$. We consider the new parameter matrix $\wedge(A)$ which has the following form:

$$\wedge(A)=\Gamma(A)+Af(\Gamma, A)\,, \tag{2}$$

where $f(\Gamma, A)$ is a quantity depending on the parameter matrix Γ and A. We assume for the sake of simplicity that $f(\Gamma, A)$, considered as a function of Γ, is regular analytic in the neighbourhood of $\Gamma=0$. As Γ is homogeneous of the first dimension with respect to A, $f(\Gamma(A), A)$ must be homogeneous of zero-th dimension with respect to A. In order that the parameter matrix $\wedge(A)$ may be invariant under transformation (1), it is necessary and sufficient that

$$\bar{\Gamma}(A)+Af(\bar{\Gamma}, A)=\Gamma(A)+Af(\Gamma, A)\,,$$

that is

$$\varphi+f(\Gamma+A\varphi, A)=f(\Gamma, A)\,, \tag{3}$$

for any value of φ. From (3) it follows that $f(\Gamma+A\varphi, A)$ must be a linear function of φ, and that the function $f(\Gamma, A)$ must satisfy the differential equation:

$$\frac{d}{d\varphi}f(\Gamma+A\varphi, A)\equiv\frac{\partial f(\Gamma, A)}{\partial\Gamma}\cdot A=-1\,. \tag{4}$$

Putting $\Gamma=0$ we moreover get from (3)

$$f(A\varphi, A)=-\varphi+f(0, A)\,,$$

where $f(0, A)$ can be an arbitrary but homogeneous function of zero-th dimension with respect to A and is independent of Γ. For this reason we may now put $f(0, A)=0$ without loss of generality. Then the function $f(\Gamma, A)$ should satisfy the functional equation:

$$\begin{aligned} &f(A\varphi, A)=\varphi f(A, A)\,,\\ &f(\Gamma+A\varphi, A)=f(\Gamma, A)+f(A\varphi, A)\,. \end{aligned} \tag{5}$$

Let $\psi(\Gamma, A)$ be an arbitrary but linear homogeneous function with respect to Γ, and $\psi(A, A)\neq 0$, then the general solution of (5) has the form

$$f(\Gamma, A)=\Phi(\Pi, A)-\frac{\psi(\Gamma, A)}{\psi(A, A)}\,, \tag{6}$$

where $\Phi(\Pi, A)$ is an arbitrary function but to satisfy the relations

1) A. Kawaguchi: Displacements in a manifold of matrices, I, Proc. **11** (1935), 39–42.

$$\Phi(0, A)=0\,, \qquad \Phi(\rho^2\Pi, \rho A)=\Phi(\Pi, A)\,,$$

and

$$\Pi=((\Gamma^i_j A^k_l-\Gamma^k_l A^i_j))\,,$$

being $\Gamma=((\Gamma^i_j))$, $A=((A^k_l))$.

2. The parameter matrix $\wedge(A)$ determined by such $f(\Gamma, A)$ not only satisfies the relation

$$f(\wedge, A)=0\,, \tag{7}$$

but also is characterized by (7). For we have from (3)

$$f(\wedge, A)=f(\Gamma+Af, A)=f(\Gamma, A)-f\,.$$

It is to be noticed that there exist also some invariant parameter matrices for (1) not of the form (2). For example, the parameter matrix

$$L(A)=\Gamma(A)-A\Gamma(A)A^{-1}\,[1] \tag{8}$$

is invariant for (1) but not of the form (2). From (8) it follows

$$A^{-1}\cdot L=0 \qquad \text{and} \qquad L\cdot A=0\,. \tag{9}$$

The parameter matrices characterized by the relations equivalent to (9) are

$$\begin{aligned} \wedge_1(A)&=\Gamma(A)-(A^{-1}\cdot\Gamma)A\,,\\ \wedge_2(A)&=\Gamma(A)-(\Gamma\cdot A)A \end{aligned} \tag{10}$$

respectively. It has meaning therefore only in the family of parameter matrices having the form (2), that (7) is a characteristic property of the parameter matrix $\wedge(A)$.

3. Let us now restrict ourselves to the special case, where $\Phi(\Pi, A)\equiv 0$, then it must be

$$\psi(\Gamma, A)=R(A)\cdot\Gamma\,, \qquad \text{whence} \tag{11}$$

$$f(\Gamma, A)=-\frac{R(A)\cdot\Gamma}{R(A)\cdot A}\,. \tag{12}$$

As $f(\Gamma, A)$ must be homogeneous of zero-th dimension with respect to A, $R(A)$ is in general homogeneous of m-th dimension:

$$\begin{aligned} R_A\cdot A&=mR\,, \qquad & \psi_A(A, A)\cdot A&=(m+1)\psi(A, A)\,,\\ \wedge_A\cdot A&=\wedge\,, & f_A(\Gamma(A), A)\cdot A&=0\,. \end{aligned} \tag{13}$$

4. Let A denote a matrixor under the group of matrix transformations G:

$$VAW=\bar{A}\,, \tag{14}$$

and let the covariant differential ∇A be also a matrixor, then it will be shown that the parameter matrix Γ is tranformed as follows:

$$\bar{\Gamma}=V\Gamma W-dVAW-VAdW\,.[2] \tag{15}$$

From (15) we can get the transformation formula of the invariant parameter matrix $\wedge(A)$:

$$\bar{\wedge}=V\wedge W-dVAW-VAdW+(\bar{f}-f)VAW\,,$$

where

$$\begin{aligned} \bar{f}&=-\frac{R(VAW)}{R(VAW)\cdot(VAW)}\cdot(V\Gamma W-dVAW-VAdW)\\ &=f-\frac{R(VAW)}{R(VAW)\cdot(VAW)}\cdot(V\wedge W-dVAW-VAdW)\,, \end{aligned}$$

as $\Gamma=\wedge-fA$. Hence

1) We assume here that A has an inverse matrix.

2) Loc. cit.

$$(16)\quad \bar{\Lambda}=(VAW-dVAW-VAdW) \\ -\frac{R(VAW)}{R(VAW)\cdot(VAW)}\cdot(V\Lambda W-dVAW-VAdW)VAW.$$

Especially, when $R(A)\cdot A$ is invariant for G, it follows from $R(A)\cdot\Lambda=\psi(\Lambda)=0$ that

(17)

$$\bar{\Lambda}=(VAW-dVAW-VAdW) \\ +\frac{1}{R(A)\cdot A}\{(AR(A))\cdot(V^{-1}dV)+(R(A)A)\cdot(dWW^{-1})\}VAW.$$

5. In conclusion we add some examples.

Ex. 1. $\psi(\Gamma, A)=A^{-1}\cdot\Gamma,\qquad f=-\frac{1}{n}A^{-1}\cdot\Gamma,$

where n denotes the order of the matrix A. The parameter matrix

$$\Lambda=\Gamma-\frac{1}{n}(A^{-1}\cdot\Gamma)A$$

is characterized by $A^{-1}\cdot\Lambda=0$. The transformation formula of the parameter matrix is given by

$$\bar{\Lambda}=(V\Lambda W-dVAW-VAdW)+\frac{1}{n}(V^{-1}\cdot dV+dW\cdot W^{-1})VAW,$$

from which follows

$$\dot{\bar{\Lambda}}=\Lambda\cdot(WV)-A\cdot d(WV)+\frac{1}{n}(V^{-1}\cdot dV+dW\cdot W^{-1})A\cdot(WV),$$

whence the norm $\dot{\Lambda}$ of the parameter matrix is invariant for $VW=E$.

Ex. 2. $\psi(\Gamma, A)=A\cdot\Gamma,\qquad f(\Gamma, A)=-\frac{1}{A\cdot A}A\cdot\Gamma,$

$$\Lambda=\Gamma-\frac{1}{A\cdot A}(A\cdot\Gamma)A,$$

whose characteristic property is $A\cdot\Lambda=0$. (16) reduces here to

$$\bar{\Lambda}=V\Lambda W-dVAW-VAdW \\ -\frac{AWV}{(AWV)\cdot(AWV)}\cdot\{\Lambda WV-A(WdV+dWV)\}VAW,$$

which is for $WV=E$

$$\bar{\Lambda}=V\Lambda W-dVAW-VAdW.$$

Ex. 3. $\psi(\Gamma, A)=\dot{\Gamma},\qquad f(\Gamma, A)=-\frac{\dot{\Gamma}}{\dot{A}},$

$$\Lambda=\Gamma-\frac{\dot{\Gamma}}{\dot{A}}A,$$

whose characteristic property is $\dot{\Lambda}=0$ and whose transformation formula is

$$\bar{\Lambda}=V\Lambda W-dVAW-VAdW \\ -\frac{\Lambda\cdot(WV)-A\cdot d(WV)}{A\cdot(WV)}VAW.$$

For $WV=E$ it becomes

$$\bar{\Lambda}=V\Lambda W-dVAW-VAdW,$$

as $\dot{\Lambda}=0$.