PAPERS COMMUNICATED

21. *On the Multivalency of an Analytic Function.*

By Shin-ichi TAKAHASHI.
Mathematical Institute, Osaka Imperial University.
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Recently some sufficient conditions for the multivalency of an analytic function in a simply-connected domain are established.[1] The object of the present note is to prove two theorems to this effect.

Theorem I. Let $f(z)=z+a_2z^2+\cdots\cdots$ be analytic and meromorphic for $|z|\leqq\rho$ $(\rho>1)$ and $f(z)\neq 0$ for $z\neq 0$ $(|z|\leqq\rho)$. Then $f(z)$ is at most p-valent in $|z|<1$ if

$$|f(z)|>\frac{\rho}{\sqrt{1+(\rho-1)^{2(p+1)}}}\qquad\text{for}\qquad |z|=\rho\,.$$

This theorem has already been proved by Bieberbach for the special case $p=1$.[2]

Theorem II. Let $f(z)=z^p+a_{p+1}z^{p+1}+\cdots\cdots$ be analytic and regular for $|z|\leqq\rho$ $(\rho>1)$. Then $f(z)$ is p-valent in $|z|<1$, if

$$\left|\frac{f(z)}{z^p}\right|<\sqrt{1+\left(1-\frac{1}{\rho}\right)^{2(p+1)}}\qquad\text{for}\qquad |z|=\rho\,.$$

Proof of Theorem I. $\varphi(z)=f^{-1}(z)$ is regular for $0<|z|\leqq\rho$ and has in $z=0$ a simple pole whose residuum is equal to 1. Therefore we have

$$\varphi(z)=\frac{1}{z}+\frac{1}{2\pi i}\int\limits_{|\zeta|=\rho}\frac{\zeta\varphi(\zeta)-1}{\zeta(\zeta-z)}d\zeta\,,\qquad |z|<1\,.$$

Putting

$$[z_0, z_1, \cdots\cdots, z_p, f]=\begin{vmatrix} 1 & z_0 & z_0^2 & \cdots\cdots & z_0^{p-1} & f(z_0)\\ 1 & z_1 & z_1^2 & \cdots\cdots & z_1^{p-1} & f(z_1)\\ \cdots & \cdots & \cdots & & & \\ 1 & z_p & z_p^2 & \cdots\cdots & z_p^{p-1} & f(z_p)\end{vmatrix} : \begin{vmatrix} 1 & z_0 & z_0^2 & \cdots\cdots & z_0^{p}\\ 1 & z_1 & z_1^2 & \cdots\cdots & z_1^{p}\\ \cdots & \cdots & \cdots & & \\ 1 & z_p & z_p^2 & \cdots\cdots & z_p^{p}\end{vmatrix}$$

where $z_0, z_1, \cdots\cdots, z_p$ lie in the unit circle, we get by induction the equality

$$[z_0, z_1, \cdots\cdots, z_p, \varphi]=\frac{(-1)^p}{z_0z_1\cdots\cdots z_p}+\frac{1}{2\pi i}\int\limits_{|\zeta|=\rho}\frac{\zeta\varphi(\zeta)-1}{\zeta(\zeta-z_0)(\zeta-z_1)\cdots\cdots(\zeta-z_p)}d\zeta\,.$$

Thus $\varphi(z)$ and also $f(z)$ are at most p-valent in $|z|<1$, if

$$\left|\frac{z_0z_1\cdots\cdots z_p}{2\pi i}\int\limits_{|\zeta|=\rho}\frac{\zeta\varphi(\zeta)-1}{\zeta(\zeta-z_0)(\zeta-z_1)\cdots\cdots(\zeta-z_p)}d\zeta\right|<1$$

1) Cf. P. Montel: Sur une formule de Weierstrass, Comptes Rendus, **201** (1935), 322.

2) Bieberbach: Eine hinreichende Bedingung für schlichte Abbildungen des Einheitskreises, Crelle Journ., **157** (1927), 189.

for all $z_0, z_1, \ldots\ldots, z_p$ lying in the unit circle. Now

$$\left|\frac{z_0 z_1 \cdots\cdots z_p}{2\pi i}\int_{|\zeta|=\rho}\frac{\zeta\varphi(\zeta)-1}{\zeta(\zeta-z_0)(\zeta-z_1)\cdots\cdots(\zeta-z_p)}d\zeta\right|$$

$$\leqq\frac{1}{2\pi}\int_0^{2\pi}\frac{|\zeta\varphi(\zeta)-1|}{|\zeta-z_0|\,|\zeta-z_1|\cdots\cdots|\zeta-z_p|}d\theta\,,\qquad \zeta=\rho e^{i\theta}$$

$$\leqq\frac{1}{2\pi}\left[\int_0^{2\pi}|\zeta\varphi(\zeta)-1|^2 d\theta\int_0^{2\pi}\frac{1}{|\zeta-z_0|^2|\zeta-z_1|^2\cdots\cdots|\zeta-z_p|^2}d\theta\right]^{\frac{1}{2}}$$

$$=\frac{1}{2\pi}\left[\int_0^{2\pi}\{|\zeta\varphi(\zeta)|^2-1\}d\theta\int_0^{2\pi}\frac{1}{|\zeta-z_0|^2|\zeta-z_1|^2\cdots\cdots|\zeta-z_p|^2}d\theta\right]^{\frac{1}{2}}$$

$$\leqq\frac{1}{2\pi}\left[2\pi\left(\rho^2M^2(\rho)-1\right)\frac{2\pi}{(\rho-1)^{2(p+1)}}\right]^{\frac{1}{2}}$$

$$=\left[\left(\rho^2M^2(\rho)-1\right)\frac{1}{(\rho-1)^{2(p+1)}}\right]^{\frac{1}{2}},$$

where $\qquad |\varphi(z)|<M(\rho)\qquad$ for $\qquad |z|=\rho\,.$

Thus $f(z)$ is at most p-valent in $|z|<1$, if

$$M(\rho)<\frac{\sqrt{1+(\rho-1)^{2(p+1)}}}{\rho},$$

i.e. $\qquad |f(z)|>\dfrac{\rho}{\sqrt{1+(\rho-1)^{2(p+1)}}}\qquad$ for $\qquad |z|=\rho\,,$

and Theorem I is proved.

Theorem II can be proved just in the same way as Theorem I. It is sufficient to remark that under the hypothesis of the theorem we have

$$f(z)=z^p+\frac{1}{2\pi i}\int_{|\zeta|=\rho}\frac{f(\zeta)-\zeta^p}{\zeta-z}d\zeta\,,\qquad |z|<1$$

and therefore

$$[z_0, z_1, \ldots\ldots, z_p, f]=1+\frac{1}{2\pi i}\int_{|\zeta|=\rho}\frac{f(\zeta)-\zeta^p}{(\zeta-z_0)(\zeta-z_1)\cdots\cdots(\zeta-z_p)}d\zeta\,.$$