56. *Certain Identities in a Generalized Space.*

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1. In this paper I shall prove certain identities in a special Kawaguchi space of order m and of dimensions n, where the curve-length

$$s=\int_{t_0}^{t} F(t, x^{(0)i}, \ldots\ldots, x^{(m)i})\,dt$$

of a curve $x^i=x^i(t)$ is invariant under transformation of t. I have mentioned in a previous paper[1] the intrinsic derivations along a curve for a contravariant vector X^i in a Kawaguchi space:

$$\text{(A)}\qquad \overset{2m-a-\rho}{D_{ij}}\,(\overset{a}{E})X^j=\sum_{\beta=\rho}^{2m-a}\binom{\beta}{\rho}\overset{a}{E}_{i(\beta)j}X^{j(\beta-\rho)},$$

where $\overset{a}{E}_i$ is a Synge vector of a-th kind[2] and we put

$$\overset{a}{E}_{i(\beta)j}=\frac{\partial \overset{a}{E}_i}{\partial x^{(\beta)j}}, \qquad X^{j(\beta-a)}=\frac{d^{\beta-a}X^j}{dt^{\beta-a}},$$

$$x^{(\beta)j}=\frac{d^\beta x^j}{dt^\beta}, \qquad x^{(0)j}=x^j.$$

Put $X^i=x^{(1)i}$ in (A), then (A) gives rise to many intrinsic vectors and these vectors may not be derived algebraically from the others and the Synge vectors in general. It should be noted, however, that these vectors are nothing but the Synge vectors except some numerical constant factors, if the curve-length s is invariant under transformation of t.

2. Before proof of this fact, it must be remarked that there exist the Craig conditions

$$\text{(1)}\qquad \sum_{\beta=\rho}^{m}\binom{\beta}{\rho}x^{(\beta-\rho+1)i}F_{(\beta)i}=\delta^1_\rho F^{3)} \qquad (\rho=1, 2, \ldots\ldots, m),$$

if s remains unaltered by transformation of t. As a consequence of (1) it follows

$$\text{(2)}\qquad \sum_{\beta=\rho}^{m}\binom{\beta}{\rho}x^{(\beta-\rho+1)i}F_{(\beta)i(\gamma)j}+\binom{\gamma+\rho-1}{\rho}F_{(\gamma+\rho-1)j}=\delta^1_\rho F_{(\gamma)j}.$$

Put

$$\text{(3)}\qquad \overset{a}{A}_{(\mu)i}(\rho)\equiv\sum_{\lambda=\rho}^{m+a}\binom{\lambda}{\rho}F_{(\mu)i}{}^{\cdot(a)}{}_{\cdot(\lambda)j}\,x^{(\lambda-\rho+1)j} \quad \text{for} \quad \rho\geqq 1,$$

then we have on account of (2)

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1) A. Kawaguchi: Some intrinsic derivations in a generalized space, Proc. **12** (1936), 149–151.

2) See A. Kawaguchi: loc. cit.

3) H. V. Craig: On a generalized tangent vector, American Journal of Mathematics, **57** (1935), 457–462. δ^a_ρ denotes the Kronecker delta, i. e. $=1$ for $a=\rho$ and $=0$ for $a\neq\rho$.

$$(4)\qquad \overset{0}{A}_{(\mu)i}(\rho)\equiv\sum_{\lambda=\rho}^{m}\binom{\lambda}{\rho}F_{(\mu)i(\lambda)j}x^{(\lambda-\rho+1)j}=\delta^1_\rho F_{(\mu)i}-\binom{\mu+\rho-1}{\rho}F_{(\mu+\rho-1)i}\,.$$

By differentiation and use of the relations

$$\Phi^{(1)}_{\cdot(\lambda)i}=\Phi_{(\lambda)i}^{\cdot\,\cdot(1)}+\Phi_{(\lambda-1)i}$$

we see from (3)

$$\overset{a+1}{A}_{(\mu)i}(\rho)=\overset{a}{A}_{(\mu)i}(\rho)^{(1)}-\sum_{\lambda=\rho}^{m+a}\binom{\lambda}{\rho}F_{(\mu)i}^{\cdot\,\cdot(a)}{}_{\cdot(\lambda)j}x^{(\lambda-\rho+2)j}+\sum_{\lambda=\rho}^{m+a+1}\binom{\lambda}{\rho}F_{(\mu)i}^{\cdot\,\cdot(a)}{}_{\cdot(\lambda-1)j}x^{(\lambda-\rho+1)j}$$

$$=\overset{a}{A}_{(\mu)i}(\rho)^{(1)}+\delta^1_\rho F_{(\mu)i}^{\cdot\,\cdot(a+1)}+\left[\sum_{\lambda=\rho}^{m+a+1}\binom{\lambda-1}{\rho-1}F_{(\mu)i}^{\cdot\,\cdot(a)}{}_{\cdot(\lambda-1)j}x^{(\lambda-\rho+1)j}\right]_{(\rho>1)},$$

so that

$$(5)\qquad \overset{a+1}{A}_{(\mu)i}(\rho)=\overset{a}{A}_{(\mu)i}(\rho)^{(1)}+\delta^1_\rho F_{(\mu)i}^{\cdot\,\cdot(a+1)}+\left[\overset{a}{A}_{(\mu)i}(\rho-1)\right]_{(\rho>1)},$$

where $[\Psi]_{(\rho>a)}$ means that the term Ψ has a meaning only for $\rho>a$ and vanishes for $\rho\leqq a$. Now we shall prove the following relations by mathematical induction

$$(6)\qquad \overset{a}{A}_{(\mu)i}(\rho)=\sum_{\beta=0}^{a}\binom{a+1}{\beta}\delta_\rho^{a-\beta+1}F_{(\mu)i}^{\cdot\,\cdot(\beta)}-\left[\sum_{\beta=0}^{a}\binom{a}{\beta}\binom{\mu+\rho-\beta-1}{\rho-\beta}F_{(\mu+\rho-\beta-1)i}^{\cdot\,\cdot(a-\beta)}\right]_{(\rho>\beta)}$$

$$a=0,1,2,\ldots\ldots,m\,.$$

It is evident from (4) that these relations hold good for $a=0$. Let us assume (6) for $a\leqq p$, then we have by differentiation

$$\overset{p+1}{A}_{(\mu)i}(\rho)=\delta^1_\rho F_{(\mu)i}^{\cdot\,\cdot(p+1)}$$

$$+\sum_{\beta=0}^{p}\binom{p+1}{\beta}\delta_\rho^{p-\beta+1}F_{(\mu)i}^{\cdot\,\cdot(\beta+1)}-\left[\sum_{\beta=0}^{p}\binom{p}{\beta}\binom{\mu+\rho-\beta-1}{\rho-\beta}F_{(\mu+\rho-\beta-1)i}^{\cdot\,\cdot(p-\beta+1)}\right]_{(\rho>\beta)}$$

$$+\sum_{\beta=0}^{p}\binom{p+1}{\beta}\delta_{\rho-1}^{p-\beta+1}F_{(\mu)i}^{\cdot\,\cdot(\beta)}-\left[\sum_{\beta=0}^{p}\binom{p}{\beta}\binom{\mu+\rho-\beta-2}{\rho-\beta-1}F_{(\mu+\rho-\beta-2)i}^{\cdot\,\cdot(p-\beta)}\right]_{(\rho>\beta+1)}$$

$$=\delta^1_\rho F_{(\mu)i}^{\cdot\,\cdot(p+1)}+\sum_{\beta=1}^{p+1}\binom{p+1}{\beta-1}\delta_\rho^{p-\beta+2}F_{(\mu)i}^{\cdot\,\cdot(\beta)}+\sum_{\beta=0}^{p}\binom{p+1}{\beta}\delta_\rho^{p-\beta+2}F_{(\mu)i}^{\cdot\,\cdot(\beta)}$$

$$-\left[\sum_{\beta=0}^{p}\binom{p}{\beta}\binom{\mu+\rho-\beta-1}{\rho-\beta}F_{(\mu+\rho-\beta-1)i}^{\cdot\,\cdot(p-\beta+1)}\right.$$

$$\left.+\sum_{\beta=1}^{p+1}\binom{p}{\beta-1}\binom{\mu+\rho-\beta-1}{\rho-\beta}F_{(\mu+\rho-\beta-1)i}^{\cdot\,\cdot(p-\beta+1)}\right]_{(\rho>\beta)}$$

$$=\sum_{\beta=0}^{p+1}\binom{p+2}{\beta}\delta_\rho^{p-\beta+2}F_{(\mu)i}^{\cdot\,\cdot(\beta)}-\left[\sum_{\beta=0}^{p+1}\binom{p+1}{\beta}\binom{\mu+\rho-\beta-1}{\rho-\beta}F_{(\mu+\rho-\beta-1)i}^{\cdot\,\cdot(p-\beta+1)}\right]_{(\rho>\beta)},$$

that is, the relation (6) holds good for $a=p+1$.

3. Now we shall proceed to prove the following theorem, which is the object of this paper:

Theorem. *If the curve-length s in a Kawaguchi space is invariant under transformation of t, then there exist the relations*

$$(\mathrm{B})\qquad \overset{2m-p-\rho}{D_{ij}}\,(\overset{p}{E})x^{(1)j}=\delta^1_\rho\overset{p}{E}_i-\sum_{l=1}^{m-p+1}\delta^l_\rho\binom{p+l-1}{l}\overset{p+l-1}{E_i}\,.$$

Proof. According to (6) we have

$$\overset{2m-p-\rho}{D_{ij}}\,(\overset{p}{E})x^{(1)j}=\sum_{a=p}^{m}(-1)^a\binom{a}{p}\sum_{\beta=0}^{m+a-p}\binom{\beta}{\rho}x^{(\beta-\rho+1)j}F_{(a)i}^{\cdot\,\cdot(a-p)}{}_{\cdot(\beta)j}$$

$$= \sum_{a=p}^{m} (-1)^a \binom{a}{p} \overset{a-p}{A}_{(a)i}(\rho)$$

$$= \sum_{a=p}^{m} (-1)^a \binom{a}{p} \sum_{\beta=0}^{a-p} \Bigg\{ \binom{a-p+1}{\beta} \delta_\rho^{a-p-\beta+1} F_{(a)i}^{\cdot \ \cdot(\beta)} - \Big[\binom{a-p}{\beta} \binom{a+\rho-\beta-1}{\rho-\beta} F_{(a+\rho-\beta-1)i}^{\cdot \ \cdot(a-p-\beta)} \Big]_{(\rho>\beta)} \Bigg\}$$

$$= \delta_\rho^1 (1-p) \sum_{a=p}^{m} (-1)^a \binom{a}{p} F_{(a)i}^{\cdot \ \cdot(a-p)} + P_i - Q_i ,$$

where

$$P_i = \Big[\sum_{a=p}^{m} (-1)^a \binom{a}{p} \sum_{\beta=0}^{a-p} \binom{a-p+1}{\beta} \delta_\rho^{a-p-\beta+1} F_{(a)i}^{\cdot \ \cdot(\beta)} \Big]_{(\rho>1)}$$

$$= \sum_{a=p}^{m} (-1)^a \binom{a}{p} \sum_{\gamma=p+1}^{a} \binom{a-p+1}{a-\gamma} \delta_\rho^{\gamma-p+1} F_{(a)i}^{\cdot \ \cdot(a-\gamma)}$$

$$= \sum_{l=2}^{m-p+1} \delta_\rho^l \sum_{a=p+l-1}^{m} (-1)^a \binom{a}{p} \binom{a-p+1}{a-l-p+1} F_{(a)i}^{\cdot \ \cdot(a-p-l+1)}$$

$$= \sum_{l=2}^{m-p+1} \delta_\rho^l \sum_{a=p+l-1}^{m} (-1)^a \binom{a}{p+l-1} \binom{p+l}{l} \frac{a-p+1}{p+l} F_{(a)i}^{\cdot \ \cdot(a-p-l+1)}$$

and

$$Q_i = \Big[\sum_{\beta=0}^{m-p} \sum_{a=p}^{m} (-1)^a \binom{a}{p} \binom{a-p}{\beta} \binom{a+\rho-\beta-1}{\rho-\beta} F_{(a+\rho-\beta-1)i}^{\cdot \ \cdot(a-p-\beta)} \Big]_{(\rho>\beta,1)}$$

$$= \sum_{l=2}^{m-p+1} \delta_\rho^l \sum_{\beta=0}^{m-p} \sum_{\gamma=p+l-\beta-1}^{m} (-1)^{\gamma-l+\beta+1} \binom{\gamma-l+\beta+1}{p} \binom{\gamma-p-l+\beta+1}{\beta} \binom{\gamma}{l-\beta} \times F_{(\gamma)i}^{\cdot \ \cdot(\gamma-p-l+1)}$$

$$= \sum_{l=2}^{m-p+1} \delta_\rho^l \sum_{\gamma=p+l-1}^{m} (-1)^{\gamma-l+1} \frac{\gamma!}{(\gamma-p-l+1)!\, p!\, l!} \times \Big\{ \sum_{\beta=0}^{l-1} (-1)^\beta (\gamma-l+\beta+1) \binom{l}{\beta} \Big\} F_{(\gamma)i}^{\cdot \ \cdot(\gamma-p-l+1)}$$

$$= \sum_{l=2}^{m-p+1} \delta_\rho^l \sum_{\gamma=p+l-1}^{m} (-1)^\gamma \binom{\gamma}{p+l-1} \binom{p+l}{l} \frac{\gamma+1}{p+l} F_{(\gamma)i}^{\cdot \ \cdot(\gamma-p-l+1)} ,$$

since

$$\sum_{a=0}^{l-1} (-1)^a \binom{l}{a} = (-1)^{l+1} , \qquad \sum_{a=0}^{l-1} (-1)^a a \binom{l}{a} = l(-1)^{l+1} .$$

Therefore we have

$$\overset{2m-p-\rho}{D}_{ij} (\overset{p}{E}) x^{(1)j} = -(p-1) \delta_\rho^1 \overset{p}{E}_i - \sum_{l=2}^{m-p+1} \delta_\rho^l \binom{p+l-1}{l} \overset{p+l-1}{E}_i ,$$

whence the theorem follows.

Corollary. *For $\tau < m-1$ there exist the relations*

$$\overset{\tau}{D}_{ij} (\overset{p}{E}) x^{(1)j} = 0 .$$

For example, it is always for $m > 2$

$$\overset{1}{D}_{ij} (\overset{p}{E}) x^{(1)j} = (2m-p) \overset{p}{E}_{i(2m-p)j} x^{(2)j} + \overset{p}{E}_{i(2m-p-1)j} x^{(1)j} = 0 .$$

4. Consider a tensor

$$g_{ik} = F F_{(m)i(m)k} + \overset{1}{E}_i \overset{1}{E}_k , \tag{7}$$

whose determinant does not vanish identically in general and put

(8) $$\overset{a}{E}{}^{k}=g^{ik}\overset{a}{E}_i\,,$$

where

(9) $$g_{ik}g^{kj}=\delta^j_i\,,$$

then we have the intrinsic derivation

(C) $$\overset{2m-1-\rho}{D}{}^{i}_{\cdot j}(\overset{a}{E})X^j=\sum_{\beta=\rho}^{2m-1}\binom{\beta}{\rho}\overset{a}{E}{}^{i}_{\cdot(\beta)j}X^{j(\beta-\rho)}\,.$$

In conclusion we shall calculate $\overset{2m-1-\rho}{D}{}^{i}_{\cdot j}(\overset{a}{E})x^{(1)j}$. For this purpose, differentiating (9), we have

$$\sum_{\beta=\rho}^{2m-1}\binom{\beta}{\rho}\{g_{ik(\beta)l}g^{kj}+g_{ik}g^{kj}_{\cdot\cdot(\beta)l}\}x^{(1)l}=0\,.$$

In another hand we can deduce from (7) and (B)

$$\sum_{\beta=\rho}^{2m-1}\binom{\beta}{\rho}g_{ik(\beta)l}x^{(1)l}=2(1-m)\delta^1_\rho FF_{(m)i(m)k}-\sum_{l=2}^{m}\delta^l_\rho(\overset{l}{E}_i\overset{1}{E}_j+\overset{l}{E}_j\overset{1}{E}_i)\,,$$

accordingly

$$\sum_{\beta=\rho}^{2m-1}\binom{\beta}{\rho}g^{kj}_{\cdot\cdot(\beta)l}x^{(1)l}=g^{ki}g^{jh}\{2(m-1)\delta^1_\rho FF_{(m)i(m)k}+\sum_{l=2}^{m}\delta^l_\rho(\overset{l}{E}_i\overset{1}{E}_j+\overset{l}{E}_j\overset{1}{E}_i)\}\,.$$

Therefore we have

$$\begin{aligned}\overset{2m-1-\rho}{D}{}^{i}_{\cdot j}(\overset{a}{E})x^{(1)j}&=\sum_{\beta=\rho}^{2m-1}\binom{\beta}{\rho}\{\overset{a}{E}_{k(\beta)j}g^{ki}+\overset{a}{E}_k g^{ki}_{\cdot\cdot(\beta)j}\}x^{(\beta-\rho+1)j}\\&=\delta^1_\rho\overset{a}{E}{}^i-\sum_{l=1}^{m-a+1}\delta^l_\rho\binom{a+l-1}{l}\overset{a+l-1}{E}{}^i\\&\qquad+\overset{a}{E}{}^k\{2(m-1)\delta^1_\rho(\delta^i_k-F\overset{1}{E}_k x^{(1)i})+\sum_{l=2}^{m}\delta^l_\rho(\overset{l}{E}{}^i\overset{1}{E}_k+\overset{l}{E}_k\overset{1}{E}{}^i)\}\\&=(2m-a)\delta^1_\rho\overset{a}{E}{}^i-2(m-1)\delta^1_\rho F^2\overset{a}{E}_k x^{(1)i}x^{(1)k}\\&\qquad-\sum_{l=1}^{m-a+1}\delta^l_\rho\binom{a+l-1}{l}\overset{a+l-1}{E}{}^i+\sum_{l=2}^{m}\delta^l_\rho\overset{a}{E}{}^k(\overset{l}{E}{}^i\overset{1}{E}_k+\overset{l}{E}_k\overset{1}{E}{}^i)\,,\end{aligned}$$

for $g_{ik}x^{(1)k}=F\overset{1}{E}_i$.