55. *Some Intrinsic Derivations in a Generalized Space.*

By Akitsugu KAWAGUCHI.

Mathematical Institute, Hokkaido Imperial University, Sapporo.

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In their very interesting papers H. V. Craig[1)] and J. L. Synge[2)] defined a Kawaguchi space and stated many intrinsic vectors and derivatives in this space. I shall introduce in this paper some another intrinsic derivations in the same space, which do not take place in the space of order one.

1. In a Kawaguchi space of order m and of dimensions n the length of a curve $x^i = x^i(t)$ $(i=1, 2, \ldots\ldots, n)$ is defined to be the invariant

$$s = \int_{t_0}^{t} F(t, x^{(0)i}, \ldots\ldots, x^{(m)i})\, dt\,,$$

where

$$x^{(a)i} = \frac{d^a x^i}{dt^a}\,, \qquad x^{(0)i} = x^i\,.$$

We shall adopt the notations

$$F_{(\beta)i} = \frac{\partial F}{\partial x^{(\beta)i}}\,, \qquad F_{(\dot\beta)i}^{\cdot\cdot(a)} = \frac{d^a}{dt^a}\frac{\partial F}{\partial x^{(\beta)i}}\,, \tag{1}$$

then it was proved by Synge[2)] that

$$\overset{a}{E}_i = \sum_{\beta=0}^{m}(-1)^{\beta}\binom{\beta}{a} F_{(\dot\beta)i}^{\cdot\cdot(\beta-a)} \qquad (a=0, 1, 2, \ldots\ldots, m) \tag{2}$$

are the components of a covariant vector, which we shall call the Synge vector of a-th kind. The Synge vector of zeroth kind is the Euler vector.

2. Let T_i be any covariant vector[3)] of order p, i. e. depended upon t, $x^{(0)i}, \ldots\ldots, x^{(p)i}$ and X^i a contravariant vector of any order, then we have the following covariant derivation along a curve for the vector X^i refered to the vector T_i.

Theorem. *The n quantities*

$$\overset{p-\rho}{D}_{ij}(T)X^j = \sum_{a=\rho}^{p}\binom{a}{\rho} T_{i(a)j} X^{j(a-\rho)} \qquad (\rho=1, 2, \ldots\ldots, p) \tag{I}$$

are the components of a covariant vector.

Proof. A point transformation $y^i = y^i(x^j)$ gives rise to the relations

$$\frac{\partial y^{(a)i}}{\partial x^{(\beta)j}} = \binom{a}{\beta} P_j^{i(a-\beta)}\,, \tag{3}$$

where

$$P_j^i = \frac{\partial y^i}{\partial x^j}\,, \qquad Q_j^i = \frac{\partial x^i}{\partial y^j}\,.$$

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1) H. V. Craig: On a generalized tangent vector, American Journal of Mathematics, **57** (1935), 456–462.

2) J. L. Synge: Some intrinsic and derived vectors in a Kawaguchi space, ibid., 679–691.

3) We can prove the analogous result, taking a contravariant vector T^i instead of the covariant vector T_i.

From (3) it follows

$$\overline{T}_{i(a)j}=Q_i^k\sum_{\beta=a}^{p}T_{k(\beta)l}\frac{\partial x^{(\beta)l}}{\partial y^{(a)j}}=Q_i^k\sum_{\beta=a}^{p}T_{k(\beta)l}\binom{\beta}{a}Q_j^{l(\beta-a)}\qquad(a>0).$$

$\overline{T}_i$ denotes the transformed T_i and $\overline{T}_{i(\beta)j}=\partial\overline{T}_i/\partial y^{(\beta)j}$. In another hand we have

$$\overline{X}^{j(a-\rho)}=(P_k^jX^k)^{(a-\rho)}=\sum_{\gamma=0}^{a-\rho}\binom{a-\rho}{\gamma}P_k^{j(a-\rho-\gamma)}X^{k(\gamma)},$$

hence

$$\begin{aligned}\overset{p-\rho}{D}_{ij}(\overline{T})\overline{X}^j&=Q_i^k\sum_{a=\rho}^{p}\sum_{\beta=a}^{p}\sum_{\gamma=0}^{a-\rho}\binom{a}{\rho}\binom{\beta}{a}\binom{a-\rho}{\gamma}Q_j^{l(\beta-a)}P_h^{j(a-\rho-\gamma)}T_{k(\beta)l}X^{h(\gamma)}\\&=Q_i^k\sum_{\beta=\rho}^{p}\sum_{\gamma=0}^{\beta-\rho}\sum_{a=\rho+\gamma}^{\beta}\binom{\beta}{\gamma}\binom{\beta-\gamma}{\rho}\binom{\beta-\rho-\gamma}{\beta-a}Q_j^{l(\beta-a)}P_h^{j(a-\rho-\gamma)}T_{k(\beta)l}X^{h(\gamma)}\\&=Q_i^k\sum_{\beta=\rho}^{p}\sum_{\gamma=0}^{\beta-\rho}\binom{\beta}{\gamma}\binom{\beta-\gamma}{\rho}T_{k(\beta)l}X^{h(\gamma)}\sum_{\delta=0}^{\beta-\rho-\gamma}\binom{\beta-\rho-\gamma}{\delta}Q_j^{l(\delta)}P_h^{j(\beta-\rho-\gamma-\delta)}\\&=Q_i^k\sum_{\beta=\rho}^{p}\binom{\beta}{\rho}T_{k(\beta)h}X^{h(\beta-\rho)}\\&=Q_i^k\overset{p-\rho}{D}_{kj}(T)X^j,\end{aligned}$$

since

$$0=\frac{d^\tau}{dt^\tau}(Q_j^iP_k^j)=\sum_{\delta=0}^{\tau}\binom{\tau}{\delta}Q_j^{i(\delta)}P_k^{j(\tau-\delta)}\quad\text{for}\quad\tau\geqq1.$$

Thus the theorem is proved.

For $\rho=p$ (I) gives us

$$\overset{0}{D}_{ij}(T)X^j=T_{i(p)j}X^j,$$

whose covariant property is evident, and for $\rho=p-1$

$$\overset{1}{D}_{ij}(T)X^j=pT_{i(p)j}\frac{dX^j}{dt}+T_{i(p-1)j}X^j,$$

which may be of interest.

3. Employing any one Synge vector, for example, of a-th kind, whose order is $2m-a$, as the vector T_i, we have the intrinsic derivation along a curve for the vector X^i:

$$\text{(II)}\qquad\overset{2m-a-\rho}{D_{ij}}(\overset{a}{E})^j=\sum_{\beta=\rho}^{2m-a}\binom{\beta}{\rho}\overset{a}{E}_{i(\beta)j}X^{j(\beta-\rho)}.$$

Thus we have some new vectors from a vector by these derivations and in general there are no algebraic relations among them. If one needs the contravariant derived vectors, the tensor

$$g_{ik}=FF_{(m)i(m)k}+\overset{1}{E}_i\overset{1}{E}_k,$$

whose determinant does not vanish identically in general although $\int Fdt$ would be invariant under transformation of t,[1] enables us to get those ones, in fact

$$g^{ik}\overset{2m-a-\rho}{D_{ij}}(\overset{a}{E})X^j\quad\text{or}\quad\overset{2m-1-\rho}{D_{\cdot j}^{i}}(\overset{a}{E}^*)X^j$$

1) See H. V. Craig: loc. cit., p. 461. He put $g_{ik}=F_{(m)i(m)k}+\overset{1}{E}_i\overset{1}{E}_k$.

are the components of the contravariant derived vectors, where

$$g^{ik}g_{kl}=\delta^i_l \quad \text{and} \quad \overset{a}{E}{}^{*i}=g^{ik}\overset{a}{E}_k .$$

4. As $\overset{1}{E}_j X^j$ is an invariant,

$$\text{(III)} \qquad \overset{2m-a-\rho}{\Delta_{ij}} X^j=\overset{1}{E}_i(\overset{1}{E}_j X^j)^{(2m-a-\rho)}$$

are the components of a vector. Since the coefficients of the highest derivatives $X^{j(2m-a-\rho)}$ in the vector $(F \overset{2m-a-\rho}{D_{ij}}(\overset{a}{E})+\overset{2m-a-\rho}{\Delta_{ij}})X^j$ are nothing but g_{ij}, we have

$$\text{(IV)} \qquad \overset{2m-a-\rho}{\delta}(\overset{a}{E})X^k=g^{ki}(F \overset{2m-a-\rho}{D_{ij}}(\overset{a}{E})+\overset{2m-a-\rho}{\Delta_{ij}})X^j$$
$$=X^{k(2m-a-\rho)}+\overset{2m-a-\rho}{\Gamma_k}(X),$$

where $\overset{2m-a-\rho}{\Gamma_k}(X)$ do not contain the highest derivatives $X^{j(2m-a-\rho)}$ and are linear with regard to X^j and their another derivatives. Especially for $2m-a-\rho=1$ the last equations reduce to

$$\text{(V)} \qquad \overset{1}{\delta}(\overset{a}{E})X^k=\frac{dX^k}{dt}+\Gamma^k_j(\overset{a}{E})X^j ,$$

where $\Gamma^k_j(\overset{a}{E})$ are independent of X^j and have the form

$$\Gamma^k_j(\overset{a}{E})=g^{ki}(F\overset{a}{E}_{i(2m-a-1)j}+\overset{1}{E}_i\overset{1}{E}{}^{\cdot(1)}_j)$$
$$=(-1)^m\binom{m}{a}Fg^{ki}\{F_{(m)i(m-1)j}+(m-a)F_{\dot{(m)}\dot{i}\dot{(m)}\dot{j}}{}^{\cdot(1)}\}+x^{(1)k}\overset{1}{E}{}^{\cdot(1)}_j ,$$

whose order is at most $2m$.

On account of (V) we can define a parallelism along a curve, i. e. two consecutive vectors X^i and X^i+dX^i are parallel, if the equations

$$\frac{dX^k}{dt}+\Gamma^k_j(\overset{a}{E})X^j=0$$

hold good.

5. In conclusion we shall apply the covariant derivations (I) to a geometry of generalized path

$$\Phi^i\equiv x^{i(p)}+\varphi^i(t, x^{(0)j}, \ldots\ldots, x^{(p-1)j})=0 .$$

As the left-hand side Φ^i of this equation has vector property, we may take it as the contravariant vector T^i, then we have from (I)

$$\overset{p-\rho}{D}{}^i_{\cdot j}(\Phi)X^j=\binom{p}{\rho}X^{i(p-\rho)}+\sum_{a=\rho}^{p-1}\binom{a}{\rho}\varphi^i_{\cdot(a)j}X^{j(a-\rho)}$$

or dividing with a constant $\binom{p}{\rho}$

$$\text{(VI)} \qquad \overset{p-\rho}{\Delta}{}^i_{\cdot j}(\Phi)X^j=X^{i(p-\rho)}+\binom{p}{\rho}^{-1}\sum_{a=\rho}^{p-1}\binom{a}{\rho}\varphi^i_{\cdot(a)j}X^{j(a-\rho)} .$$

Specially for $\rho=p-1$ (VI) becomes

$$\overset{1}{\Delta}{}^i_{\cdot j}(\Phi)X^j=\frac{dX^i}{dt}+\frac{1}{p}\varphi^i_{\cdot(p-1)j}X^j ,$$

which was mentioned by D. D. Kosambi[1]. already.

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1) D. D. Kosambi: An affine calculus of variations, Proc. of the Indian Academy of Sciences, **2** (1935), 333–335.