41. *A Formulation of Operational Calculus.*

By Tosio KITAGAWA.

Mathematical Institute, Faculty of Science, Osaka Imperial University.

(Comm. by T. YOSIE, M.I.A., May 12, 1937.)

1. The object of this note is to communicate a formulation of operational calculus which is a method of solving some class of functional equations by considering allied functional operations which are permutable with certain fundamental operation as functions of the latter and thus by reducing the problems to the ordinary calculus of complex valued functions whose independent variable is real or complex number.

Our method is much suggested by a new expansion formula of Delsarte's,[1] and the results yield us a generalisation of some of our theorems on the linear translable functional equations and Cauchy's series.[2]

2. *Convections.* 1° Let us designate by small latin letters elements of an abstract space X, by large latin letters subsets[3] of X, and by large latin letters within round brackets linear set of functions defined at a subset of X.

In the following we assume that for any point t of X there correspond three sets $(A)_t$, $(B)_t$ and $(C)_t$ consisting of functions $f(x)$ defined on the set $Y_t \subset X$, and such that $(A)_t \supset (B)_t \supset (C)_t$. The class (A) is composed with all $f(x)$ which are defined over X and which belong to $(A)_t$ as functions defined on Y_t. The class (B) is composed with all $f(x)$ which belong to both (A) and the domain of the linear operation $\mathfrak{D}$.

2° The fundamental elements of $(B)_t$ are a class of functions $j_\lambda(x, t)$ which belong to $(B)_t$, and which are parametrically dependent upon λ, where λ constitutes a two dimensional simply connected domain $\mathfrak{M}$ in the complex λ-plane.

---

1) J. Delsarte: [I] Sur un principle générale de developpement des fonctions d'une variable réelle en série des fonctions entières, C. R., Paris, **200** (1935).

[II] Sur l'application d'un principle général de développement des fonctions d'une variable aux séries de fonctions de Bessel, C. R., Paris, **200** (1935).

[III] Sur un procédé de développement des founctions en séries et sur quelques applications. J. Math. pures appl., IX, s. **15** 97–102 (1936).

We shall quote these papers by I, II and III respectively.

2) T. Kitagawa: On the theory of linear translatable functional equation and Cauchy's series, Japanese Journ. Math. **13** (1937). (Under press). We shall quote this paper by $[T]$.

3) Under a subset of X, we always mean an "echte" subset.

4) For example we may mention a special case defined as follows:

1° $X=(-\infty, +\infty)$

2° $Y_t=(t-a, t+a)$ $(0 < a < +\infty)$

3° $(A)_t$ is consisted of all functions defined and quarely integrable in the interval $(t-a, t+a)$.

4° $(B)_t$ is consisted of all functions defined and m-time differentiable in the interval $(t-a, t+a)$.

5° $(C)_t$ is consisted of all functions $f(x)$ which belong to $(B)_t$ and such that $f^{(m)}(t)=0$.

$\mathfrak{D}f(x)$ is a linear operation whose domain is (B), and it possesses the spectral property that

$$\mathfrak{D}\left[j_\lambda(x,t)\right]=\varphi(\lambda)j_\lambda(x,t)\,, \qquad (\lambda\in\mathfrak{M}) \tag{2.1}$$

where we assume that both $\varphi(\lambda)$ and $j_\lambda(x,t)$ (for any fixed x on X) are uniform and regular in any domain interior to $\mathfrak{M}$.

I. *First principle of uniqueness*: for any $\lambda\in\mathfrak{M}$, the functions which belong to $(B)_t$ and which satisfy

$$\mathfrak{D}\left[f(x)\right]=\varphi(\lambda)f(x) \tag{2.2}$$

are of the form

$$f(x)=kj_\lambda(x,t)\,, \tag{2.3}$$

k being an arbitrary constant; therefore $\varphi(\lambda)$ is univalent on $\mathfrak{M}$.

II. *Second principle of uniqueness*: for any given $\lambda\in\mathfrak{M}$ and any given function $g(x)\in(A)_t$ the equation

$$\mathfrak{D}\left[f(x)\right]=\varphi(\lambda)f(x)+g(x) \tag{2.4}$$

has one and only one solution in $(C)_t$, which we will designate by $\mathfrak{L}^t_\lambda[g(x)]$.

III. The function-sets $\{j_{\lambda_1,\lambda_2,\dots,\lambda_n}(x,t)\}$ are defined by induction as follows:[1]

(i) We assume that

$$\mathfrak{L}^t_{\lambda_2}\left[j_{\lambda_1}(x,t)\right]=\begin{cases}\dfrac{j_{\lambda_2}(x,t)-j_{\lambda_1}(x,t)}{\varphi(\lambda_2)-\varphi(\lambda_1)} & [\lambda_1\neq\lambda_2]\\[2ex] \left(\dfrac{\partial j_\lambda(x,t)}{\partial\varphi(\lambda)}\right)_{\lambda=\lambda_1=\lambda_2} & [\lambda_1=\lambda_2]\,,\end{cases} \tag{2.5}$$

which we shall denote by $j_{\lambda_2,\lambda_1}(x,t)$.

(ii) Let us suppose that the functions $j_{\lambda_1,\lambda_2,\dots,\lambda_{n-1}}(x,t)$ are already defined for any set $(\lambda_1,\lambda_2,\dots,\lambda_{n-1})$ of $n-1$ complex numbers.

Then we shall assume that

$$\mathfrak{L}^t_{\lambda_n}\left[j_{\lambda_1,\lambda_2,\dots,\lambda_{n-1}}(x,t)\right]=\begin{cases}\dfrac{j_{\lambda_1,\dots,\lambda_{n-1},\lambda_n}(x,t)-j_{\lambda_1,\dots,\lambda_{n-2},\lambda_n}(x,t)}{\varphi(\lambda_n)-\varphi(\lambda_{n-1})} & [\lambda_n\neq\lambda_{n-1}]\\[2ex] \left(\dfrac{\partial j_{\lambda_1,\dots,\lambda_{n-2},\lambda}}{\partial\varphi(\lambda)}\right)_{\lambda=\lambda_{n-1}=\lambda_n} & [\lambda_{n-1}=\lambda_n]\end{cases} \tag{2.6}$$

and we shall denote them by $j_{\lambda_1,\lambda_2,\dots,\lambda_n}(x,t)$. Specially when $\lambda_1=\lambda_2=\dots=\lambda_n=\lambda$, then we use an abbreviate notation $j_{\lambda^n}(x,t)$.

3° We shall assume that a linear operation Λ, whose domain $d(\Lambda)$ and range $r(\Lambda)$ are subsets of (A), possesses the following properties:

I. For any fixed element t in X, $\Lambda f(t)$ is a linear functional whose domain is $(A)_t$, and is written in accurate form

$$\Lambda f(t)=\Lambda\left|\left[t;f(x):x\in Y_t\right]\right|. \tag{2.7}$$

1) In Delsarte's formula, these systems of functions are considered only up to $n=2$.

II. $\Lambda f(t)$ is permutable with the operation $\mathfrak{D}$ in the sense that, if $f(x)\in(B)_t\cdot d(\Lambda)$, $\mathfrak{D}f(x)\in d(\Lambda)$ and $\Lambda f(t)\in(B)_t$, then we have

$$(2.8)\qquad \Lambda\left|\left[t;\,\mathfrak{D}_x f(x):x\in Y_t\right]\right|=\mathfrak{D}_t\Lambda\left|\left[t;f(x):x\in Y_t\right]\right|$$

or simly writing,

$$(2.9)\qquad \Lambda\mathfrak{D}f(t)=\mathfrak{D}\Lambda f(t)\,.$$

3. *Introduction of Cauchy-Delsarte's series.* Our chief concerns are at first the functional equation

$$(3.1)\qquad \Lambda f(x)=0\,,\qquad (x\in Y_t)\,.$$

To solve this, we will define *a $\mathfrak{C}$-section of Cauchy-Delsarte's series of $f(x)\in(A)_t$ with respect to Λ at a point t* by a contour-integral[1]

$$(3.2)\qquad S_{\mathfrak{C}}(x,t;f)=\frac{1}{2\pi i}\oint_{\mathfrak{C}}\frac{j_\lambda(x,t)}{\Lambda j_\lambda(t,t)}\Lambda\mathfrak{L}_\lambda^t\left[f(t)\right]d\varphi(\lambda)\,.$$

This is a generalisation of what we called a section of Cauchy's series,[2] and there hold some fundamental theorems analogous to those concerning the latter. Here we will communicate the following theorems:

Theorem I.[3] *If $j_\lambda(x,t)\in d(\Lambda)$ and $\Lambda j_\lambda(x,t)\in(B)_t$ for any $\lambda\in\mathfrak{M}$, then there corresponds a function $G(\lambda)$ such that*

$$(3.3)\qquad \Lambda j_\lambda(x,t)=G\big(\varphi(\lambda)\big)j_\lambda(x,t)\,,\qquad (x\in Y_t)\,.$$

Hereafter *we assume that* $G\big(\varphi(\lambda)\big)$ is a regular function of λ in any domain interior of $\mathfrak{M}$.

Theorem II.[4] *Let $\mathfrak{C}$ contain a point μ of $\mathfrak{M}$ in its interior. If, for $1\leqq\nu\leqq k$, $j_{\mu^\nu}(x,t)$ are solutions of* (3.1), *then we have*

$$(3.4)\qquad S_{\mathfrak{C}}(x,t;j_{\mu^\nu})=j_{\mu^\nu}(x,t)\,,\qquad (1\leqq\nu\leqq k)\,.$$

Corollary I. *Let the zeros of $G\big(\varphi(\lambda)\big)$ located in $\mathfrak{M}$ be denoted by $\{\lambda_k\}$, and let the multiplicities of λ_k be m_k respectively. Then, for any system of constants $\{a_{k,\nu}\}$, the function defined by*

$$(3.5)\qquad h(x)=\sum_{k=1}^{N}\sum_{\nu=1}^{m_k}a_{k,\nu}j_{\lambda_k^\nu}(x,t)$$

is a solution of (3.1), *and if $\mathfrak{C}$ contains $\lambda_1,\lambda_2,\cdots,\lambda_{N-1}$ and λ_N in its interior, then we have*

$$(3.6)\qquad S_{\mathfrak{C}}(x,t;h)=h(x)\,.$$

Thus we see that the fundamental theorems similar to those on the

---

1) Although our method is much indebted to that of Delsarte's, it must be noticed that in the latter he considered the linear functionnal δ in stead of linear operation Λ.

2) See [T] Introduction, Definition II.

3) This is a generalisation of the formulue (0.12) in [T] Introduction.

4) This is a generalisation of the formulue (0.302) in [T] Introduction.

theory of linear translatable functional equation hold in our general formulation, except that of the most important, that is to say, expansion theorem of *arbitrary function* of $(A)_t$ into its Cauchy-series. To secure the latter with or without any restriction to the operators $\mathfrak{D}$ and Λ is worth while to investigate, and will enable us to decide whether we may be permitted to proceed in a way parallel to the theory of linear translatable functional equation or not.

Some well known theorems on the theory of linear differential equation with constant coefficients, however, may be generalised to the solutions of the equation (3.1), and to those of the non-homogeneous equation

$$\Lambda f(x) = g(x), \qquad (x \in X). \tag{3.7}$$

This results from the fact that these theorems do not appeal to any expansion-theorem of arbitrary function (which belongs to (A) and which is not necessarily a solution of the function equation (3.1)) into its Cauchy-Delsarte's series. In the near future, we shall communicate some of them.