40. *On Valiron's Theory of Linear Differential Equation of Infinite Order with Constant Coefficients.*

By Tosio KITAGAWA.

Mathematical Institute, Faculty of Science, Osaka Imperial University.

(Comm. by T. YOSIE, M.I.A., May 12, 1937.)

1. Previously[1] we have given an interpretation of the Valiron's theory of linear differential equation of infinite order with constant coefficients[2] from the standpoint of the theory of linear translatable functional equation. We wish, in the following, to complete this idea in some respect in order to bring it into a more intimate connection with the original Valiron's theory.

2. The most important theorem[3] in the Valiron's theory is the following: Let Λ be a linear differential operator of infinite order with constant coefficients whose generating function $G(\lambda)$ is an integral function of order 1 and of mean type with Valiron's condition.[4] Let $f(z)$ be a solution of the functional equation

$$\Lambda f(z)=0\,, \qquad (|z-z_0|<h) \tag{1}$$

and let it be regular in the domain $|z-z_0|<D+h$.[4']

Then a sufficient condition for that $f(x)$ is developable into its Dirichlet's series in the domain $|z-z_0|<h$ is that the series[5]

$$\sum |e^{a_n z}Q_n(z)| \tag{2}$$

should converge in the first domain $|z-z_0|<D+h$.

We will connect this theorem with our expansion-theory of Cauchy's series.[6]

3. What Ritt and Valiron called a Dirichlet series is nothing but the Cauchy's series of $f(z)$ with respect to the linear translatable operator.[7]

Its section with respect to a contour C_r is, therefore, given by

$$S_r(z, z_0; f)\equiv\frac{1}{2\pi i}\oint_{C_r}\frac{e^{\lambda z}}{G(\lambda)}\underset{0}{\Lambda_\xi}\left[e^{\lambda\xi}\int_0^\xi e^{-\lambda\eta}f(z_0+\eta)\,d\eta\right]d\lambda\,. \tag{4}$$

Now the direct computation yields us

---

1) T. Kitagawa: On the theory of linear translatable functional equation and Cauchy's series, Japan. Journ. Math., **13** (1937) (Under press).

2) G. Valiron: Sur les solutions des équations différentielles linéaires d'ordre infini et a coefficients sonstants, Annales scient. l'école norm. sup., III serie, Tome **46** (1929).

3) See G. Valiron, loc. cit., Theorem XVI (p. 41).

4)-4') See G. Valiron, loc. cit., Theorem XVI (p. 41).

5) $\sum e^{a_n z}Q_n(z)$ is the Dirichlet series of $f(z)$.

6) See T. Kitagawa, loc. cit., Introduction and Chaper II, § 9.

7) See T. Kitagawa, loc. cit., Introduction, specially Definition II.

There we have defined a Cauchy's series in the real range, but it may be easily generalised to a complex domain.

$$(5)\qquad \frac{d^n}{d\xi^n}\left[e^{\lambda\xi}\int_0^\xi e^{-\lambda\eta}f(z_0+\eta)d\eta\right]=e^{\lambda\xi}\int_0^\xi e^{-\lambda\eta}\frac{d^n f(z_0+\eta)}{d\eta^n}\,d\eta$$
$$+\left[\frac{d^n}{d\xi^n}\left\{e^{\lambda\xi}\int_0^\xi e^{-\lambda\xi}f(z_0+\eta)d\eta\right\}\right]_{\xi=0}e^{\lambda\xi}$$

and therefore, for any linear translatable operator defined by symbolic notation

$$(6)\qquad B_r\left(\frac{d}{dx}\right)\equiv e^{s\frac{d}{dx}}\prod_{i=1}^{r}\left(1-\frac{\frac{d}{dx}}{a_i}\right)$$

we have, in virtue of Valiron's theorem on transmutation,[1]

$$(7)\qquad B_r\left(\frac{d}{dx}\right)\left\{e^{\lambda\xi}\int_0^\xi e^{-\lambda\xi}f(z_0+\eta)d\eta\right\}=e^{\lambda\xi}\int_0^\xi e^{-\lambda\xi}B_r\left(\frac{d}{d\eta}\right)f(z_0+\eta)d\eta$$
$$+\left[B_r\left(\frac{d}{d\xi}\right)\left\{e^{\lambda\xi}\int_0^\xi e^{-\lambda\eta}f(z_0+\eta)d\eta\right\}\right]_{\xi=0}e^{\lambda\xi}.$$

The composition-theorem of Valiron's[2] gives us

$$(8)\qquad \underset{0}{A_\xi}\left[e^{\lambda\xi}\int_0^\xi e^{-\lambda\eta}f(z_0+\eta)d\eta\right]$$
$$=\underset{0}{B_\xi}\left[A\left(e^{\lambda\xi}\int_0^\xi e^{-\lambda\eta}f(z_0+\eta)d\eta;\,p_1,p_2,\dots,p_r\right);\,p_1,\dots,p_r\right].$$

For the sake of brevity, let us put[3]

$$(9)\qquad A[y;\,p_1,p_2,\dots,p_r]=A^r(y)$$

$$(10)\qquad B[y;\,p_1,p_2,\dots,p_r]=B^r(y)$$

and

$$(11)\qquad e^{\lambda\xi}\int_0^\xi e^{-\lambda\eta}f(z_0+\eta)d\eta=\mathfrak{L}_\lambda(f(\xi)).$$

Then the combination of (7) and (8) leads us to

$$(12)\qquad \underset{0}{A_\xi}\left[\mathfrak{L}_\lambda(f(\xi))\right]=\underset{0}{B^r_\xi}\left[\mathfrak{L}_\lambda\left(A^r(f(\xi))\right)\right]+\underset{0}{A^r_\xi}\left[\left(\mathfrak{L}_\lambda(f(\xi))\right)\right]\underset{0}{B^r_\xi}(e^{\lambda\xi}).$$

Let us select $B^r(y)$ such that a_{p_i} $(i=1,2,\dots,r)$ are identical with the zero-points of $G(\lambda)$ located in the interior of the contour $\mathfrak{C}_r$ until their multiplicities. Operating A^r_z on both sides of (4) as the functions of z, we obtain

$$(13)\qquad A^r_z\left[S_r(z,z_0;f)\right]=\frac{1}{2\pi i}\oint\frac{A^r_z(e^{\lambda z})}{G(\lambda)}\underset{0}{A_\xi}\left[e^{\lambda\xi}\int_0^\xi f(z_0+\eta)e^{-\lambda\eta}d\eta\right]d\lambda$$
$$=\frac{1}{2\pi i}\oint_{\mathfrak{C}_r}\frac{e^{\lambda z}}{\underset{0}{B^r_z}(e^{\lambda z})}\underset{0}{B^r_\xi}\left[\mathfrak{L}_\lambda\left(A^r(f(\xi))\right)\right]d\lambda$$
$$+\frac{1}{2\pi i}\oint_{\mathfrak{C}_r}\frac{e^{\lambda z}}{\underset{0}{B^r_z}(e^{\lambda z})}\underset{0}{A^r_\xi}\left[\left(\mathfrak{L}_\lambda(f(\xi))\right)\right]\underset{0}{B^r_\xi}(e^{\lambda\xi})\,d\lambda$$

---

1) See Valiron, loc. cit., Theorem VIII and IX (p. 35).

2) See Valiron, loc. cit., Theorem XI (p. 37).

3) For the definition of $A[y;\,p_1,\dots,p_n]$, $B[y;\,p_1,p_2,\dots,p_n]$, see Valiron, loc. cit., p. 36.

where the second term of the right-hand side vanishes and the first term is equal to $A_z^r(f(z))$, for $A_z^r(f(z))$ is a solution of the linear differential equation *of the finite order*, i. e.,

$$B_z^r\Big(A_z^r(f(z))\Big)=0 \tag{14}$$

and consequently, as well known, it should be identical with its Cauchy's series.

Thus we have obtained

$$A_z^r\big(S_r(z, z_0; f)\big)=A_z^r\big(f(z)\big) . \tag{15}$$

4. After these preparations, we are now in a position to give another proof of Valiron's theorem stated in § 2.

By the hypothesis there is a sequence of contours $\{\mathfrak{C}_r\}$ such that $S_r(z, z_0; f)$ tends to a limiting function $S(z, z_0; f)$ in the domain $|z-z_0|< D+h$. Since $f(z)$, $S(z, z_0; f)$ and $S_r(z, z_0; f)$ are the regular solutions of the functional equation (1), we may apply the approximation-theorem of Valiron's,[1] and then we shall obtain that

$$\lim_{r\to\infty} A_z^r\big(S_r(z, z_0; f)\big)=\lim_{r\to\infty} A_z^r\big(S(z, z_0; f)\big)=S(z, z_0; f) \tag{16}$$

and that

$$\lim_{r\to\infty} A_z^r\big(f(z)\big)=f(z) . \tag{17}$$

Consequently, in view of (15)-(17), we obtain

$$f(z)=S(z, z_0; f) ,$$

which we were to prove.

---

1) See Valiron, loc. cit., Theorem XII (p. 37).

See Valiron, loc. cit., § 5 Propriétés des solutions déduites du développement en serie de solutions fondamentales (p. 38).

Here we appeal also to the assumption that the series (2) should converge in the domain $|z-z_0|< D+h$.